Gauge Theories Unification Schemes

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1 Introduction

Reading back on these notes, I don't think I have done a good job presenting this beautiful idea of unification. I think I have delved too much into details instead of exposing the simplicity behind the subject. I hope I can remedy this in the lecture. But really the idea is very simple (without trivializing it of course). We observe the Standard Model to be a gauge theory $SU(3) \times SU(2) \times U(1)$. Why not try to embed it as a subgroup in some larger gauge group and see what happens. This would of course mean that at some energy scale the coupling constants all unite. But it implies many other things as well as we shall explore below. There are many constraints, however, on the larger gauge group. Anomalies have to cancel for example. But the moment you set your mind on trying to embed the Standard Model into a larger gauge group, it is just a matter of trying different schemes and see if they work. The tools are all available, the Simple Lie groups have all been classified and well-studied. And today experimentalists offer a lot of experimental constraints (proton decay, flavor changing neutral currents) that we can use to cross out certain schemes or put lower bounds on the unification scale. I will start this presentation by explaining why are we at all interested in just Simple Lie groups. Then in section 3 I will go on to remind you of some results from group theory that might come in handy. The Standard Model is briefly summarized in section 4 and $SU(5)$ unification is presented in section 5. I haven't discussed $SO(10)$ unification in this notes but I hope to do so in the lecture.

2 Gauge Theories

As we discussed in class, in order to construct a quantum field theory of unit spin which is manifestly Lorentz invariance and preserve unitarity we need our theory to be gauge invariant. In order to achieve this we construct our Lagrangian out of the matter fields, $\psi(x)$, the covariant derivative of the fields $D_\mu \psi(x)$ and the field strength tensor $F^a_{\mu\nu}$,

$$D_\mu \psi(x) \equiv (\partial_\mu - i A^a_\mu t^a)\psi(x) \tag{1}$$

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_{nu} \tag{2}$$

Where A^a_μ are my gauge fields (I will be suppressing the matrix indices on the generators t^a, and the field $\psi(x)$). Note that I have written the structure constant C^a_{bc} distinguishing between the superscript and the two subscripts which are anti-symmetric (since $[t_b, t_c] = iC^a_{bc} t_a$). As we will see below, for the Lie algebras we are interested in, the structure constant can be chosen to be anti-symmetric in all three indices. Under a general gauge transformation those creatures transform as,

$$\delta\psi(x) = i\epsilon^a t^a \psi(x) \tag{3}$$

$$\delta(D_\mu \psi(x)) = i\epsilon^a t^a (D_\mu \psi(x)) \tag{4}$$

$$\delta(F^a_{\mu\nu}) = i\epsilon^c C^a_{bc} F^b_{\mu\nu} \tag{5}$$

Note that the field strength tensor transform as a matter field in the adjoint representation ($(t^a)_{bc} = -iC^a_{bc}$). Constructing a Lagrangian which is invariant under a global transformation out of these quantities, ensures that it is also invariant under a local transformation. The only way we can write a kinetic term for the gauge field is by using the $F^a_{\mu\nu}$ tensor. Lorentz invariance dictates the form,

$$\mathcal{L}_A = -\frac{1}{2} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} \tag{6}$$

Where g_{ab} is a constant matrix. For parity non-conserving theories (which the Standard Model is a good example of) we can also use the totaly anti-symmetric epsilon tensor to write,

$$\mathcal{L}'_A = -\frac{1}{2} \theta_{ab} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^b_{\alpha\beta} \tag{7}$$

θ_{ab} is again a constant matrix. This can actually be written as a total derivative and therefore doesn't enter perturbation theory in any order. To see that the above term indeed violates parity just do a Lorentz transformation on the Field Tensor with $\Lambda^\mu_\nu = diag(1, -1, -1, -1)$ corresponding to a parity transformation and note that the result is proportional to $-\epsilon^{\mu\nu\alpha\beta}$ rather than $\epsilon^{\mu\nu\alpha\beta}$.
Now it turns out that g_{ab} is quite a special matrix. It can obviously be chosen symmetric (since $F^a_{\mu\nu} F^{b\mu\nu}$ is symmetric with respect to a and b), and it has to be real so that our Lagrangian is real. Since we require gauge invariance we must also have,

$$g_{ab} F^a_{\mu\nu} C^b_{cd} F^{c\mu\nu} = 0 \tag{8}$$

for all choices of d. Since we don't want to impose any constraints on our Field Tensor, we must require the matrix g_{ab} to satisfy,

$$g_{ab} C^b_{cd} = -g_{cb} C^b_{ad} \tag{9}$$

Canonical quantization and positivity of the scalar product require that g_{ab} be a positive definite matrix. So g_{ab} is a symmetric, real, positive definite matrix, which satisfy the condition (9) above. This is very important since it severely restricts the type of Lie algebras we need to consider in constructing gauge theories. As Weinberg proves in the Appendix for chapter 15 of his book, reference [4], the following three statements are equivalent,
(a) There exists a symmetric, real, positive definite matrix over the space of group indices, which satisfy the condition (9).

(b) We can always choose an appropriate basis t'^a for our generators (remember, the generators of Lie group form a real linear space of generators), such that the structure constants C^a_{bc} is anti-symmetric in all three indices. Therefore there is no point of me writing a as a superscript and from now on I'll just write C_{abc} when referring to the structure constant of the Lie algebras involved.

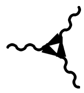
(c) (This is the thing to remember) The Lie algebra must be a direct sum of commuting **compact simple** and $U(1)$ subalgebras. Put in the language of groups rather than algebras, our Lie group G must be a direct product of compact simple and $U(1)$ subgroups.

I'll remind you below what is a *simple Lie algebra* when discussing group theory. This is very important. It highly restricts the types of Lie groups we have to consider as candidates for our gauge theories. Moreover, the nice thing (or the sad thing, depending how you look at it) is that all the relevant work has been done in the beginning of the 20^{th} century by Eli Cartan and Sophius Lie (and others). They have completely classified all the possible simple Lie groups. There are definite algorithms for computing all their irreducible representations, subgroups, generators and other goodies. I'll present this results later on.
Weinberg also shows that we can now simply choose $g_{ab} = \delta_{ab}$. If you wonder where did the coupling constants go, they are hidden in the definition of the gauge field A^a_μ and I will display them explicitly later on.

When discussing modifications of the fermion or gauge structure of the Standard Model it is important that we don't introduce anomalies (the Standard Model is anomaly-free). As Jesse explained to us last week very clearly anomalies arise when considering contributions to the vertex of three currents as shown in the diagram below. Those diagrams are linearly divergent. If the currents are associated with global symmetries of the theories, this is not very harmful. However, when those currents are associated with the gauge fields of the theory this is bad news. Those diagrams will contribute to the vertex renormalization for the gauge coupling constant. The linear divergence mean that we will not be able to regulate this vertex in a way compatible with the gauge symmetry of our theory. The anomaly coefficient is given by,

$$A^{abc} \simeq 2Tr(T_L^a T_L^b, T_L^c) - 2Tr(T_R^a T_R^b, T_R^c) = 2(A_L^{abc} - A_R^{abc}) \tag{10}$$

Where $T^a_{L,R}$ are the representation generators corresponding to the left (right) handed fermions. Therefore, in vector-like theory (non-chiral) anomalies are automatically cancelled. It turns out that of all the classical groups (described below), only the $SU(N)$'s and $SO(6)$ have anomalies that one should worry about.



3 Some results from Group theory

I would like to remind you of some results on group theory (I give a few references at the end where you can find the theory developed in a more or less systematic way). We will concentrate on *Lie groups*, groups of transformations which depend on a set of continuous variables θ_a. Those elements of the group which lay in the connected part of the group can be written as exponentials of a set of operators $U(\vec{\theta}) = \exp(-it_a\theta_a)$, which are called the generators of Lie group. The groups multiplication law forces the generators to obey certain commutation relations which are referred to as the *Lie algebra* associated with the Lie group,

$$[t_b, t_c] = iC^a_{bc} t_a \tag{11}$$

Where the C^a_{bc} are the structure constants of the Lie algebra. A representation for the Lie group can be generated by finding a representation for the Lie algebra, i.e. a set of operators satisfying the above commutation relation. A very important representation is the **adjoint representation**. It is formed by taking,

$$(T^a)_{bc} = -iC^a_{bc} \tag{12}$$

Notice the difference between a representation of the generators (the above T^a's are a representation) and the generators themselves (the t_a's, they are of course in a sense also a representation called the defining representation of the algebra). We will say that an operator (a field usually) transform according to the D representation of the Lie group if,

$$\mathcal{O} \rightarrow \mathcal{O}' = U(\theta_a)\mathcal{O}U^\dagger(\theta_a) = \exp(-i(T^{(D)})_a\theta_a)\mathcal{O} \tag{13}$$

Let's now specialize to *simple Lie algebras*. We need a few definitions. We will say that a Lie algebra has an invariant subalgebra if we can find a set of generators $\mathcal{H} = H_a$ such that the commutator of any member of the set with any member of the entire algebra is inside the set $\mathcal{H}$. In symbols $[t_a, H_b] \in \mathcal{H}$. A **simple** Lie algebra is one that has **no invariant subalgebras**. A **semi-simple** Lie algebra is one that has **no Abelian invariant subalgebras** (an Abelian algebra is one in which all the generators commute, or put it differently, one in which all the structure constants are zero). A **compact** simple or semi-simple Lie algebra is one for which $Tr(t_a^{(Adj)} t_b^{(Adj)}) = -C^c_{ad}C^d_{bc}$ is positive-definite.
The **Cartan** subalgebra of a group is the maximal set of generators that can be simultaneously diagonalized

(i.e. they all commute with each other). I will denote them by H_a, with the index $a = 1..m$. m is called the rank of the Lie algebra. So we have,

$$[H_a, H_b] = 0 \tag{14}$$

for all $a, b = 1..m$. The rank of the group tells us how many good quantum numbers we have for our state in the theory which will describe particles. That is, the different charges for our particles (electric charge, weak charge etc.).

As I mentioned before, people have classified the simple Lie groups completely long ago. The method of highest weight can be used to find all the finite dimensional representation of the compact simple Lie groups for example. There are four countable sequences of Lie algebras, A_l, B_l, C_l and D_l where l is the rank. Those correspond to the classical groups of geometrical rotations. A_l corresponds to $SU(l+1)$ the group of transformation, over a complex $l+1$ dimensional linear space, which leave the inner product $u_i^* v_i$ invariant. B_l and D_l correspond to $SO(2l+1)$ and $SO(2l)$ respectively. $SO(n)$ is of course the group of rotations over a real n dimensional linear space that leave the inner product $v_i u_i$ invariant. The last classical group C_l correspond to the group of symplectic matrices $Sp(2l)$. Those are real $2l \times 2l$ matrices, which leave the quadratic form $u_i S_{ij} v_j$ invariant, where S_{ij} is the skew-symmetric matrix,

$$S = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix} \tag{15}$$

Beside of the classical groups there are also five exceptional groups. Those are labelled G_2, F_4, E_6, E_7 and E_8 (where again, the subscript denotes the rank of the group). I have summarized in the table below all of the simple groups.

Cartan Label	Classical Group	Order(N)	Range of l
A_l	$SU(l+1)$	$l(l+2)$	$l \geq 1$
B_l	$SO(2l+1)$	$l(2l+1)$	$l \geq 2$
C_l	$Sp(2l)$	$l(2l+1)$	$l \geq 3$
D_l	$SO(2l)$	$l(2l-1)$	$l \geq 4$
G_2	G_2	14	
F_4	F_4	52	
E_6	E_6	78	
E_7	E_7	133	
E_8	E_8	248	

Note that if T_a is a representation of the generators than so is $-(T_a)^*$. This follows since,

$$[T_a, T_b] = iC_{abc}T_c \qquad \Rightarrow \qquad [(-T_a)^*, (-T_b)^*] = -iC_{abc}(T_c)^* = iC_{abc}(-T_c)^* \tag{16}$$

$-T_a^*$ is called the complex conjugate representation. However, it doesn't mean that we found a new representation. It might be that the complex conjugate representation $-T_a^*$ is unitarily equivalent to the representation T_a, in which case it is not a new representation at all. If we can find a unitary matrix U such that

$$(-T_a)^* = U(T_a)U^\dagger, \qquad a = 1..N \tag{17}$$

(note that U must be the same unitary matrix for all the generators) then the representation is called real, or when applied to fermions it is called vector-like (since we cannot rotate the right and left handed components of the field differently). If we cannot find such a unitary matrix, then the representation is said to be complex. If the left handed fermions belong to such a representation then the right handed ones will belong to the complex conjugate representation and since it is different the theory is said to be chiral (i.e. we can rotate the right and left components differently).

4 Standard Model

The minimal non-supersymmetric (I will not include supersymmetry in my talk at all, but it might be interesting to discuss GUTs in the context of SUSY at some point, maybe later in the summer) Standard Model of particle physics is an $SU(3) \times SU(2) \times U(1)$ gauge theory with the following matter content,

Particle	SU(3)	SU(2)	U(1)
q	3	2	$\frac{1}{6}$
u^c	$\bar{3}$	-	$-\frac{2}{3}$
d^c	$\bar{3}$	-	$\frac{1}{3}$
l	-	2	$-\frac{1}{2}$
e^c	-	-	1
ν^c ???	-	-	-
H	-	2	$\frac{1}{2}$

This table repeats itself three times for the three families. I have suppressed the color index on $q = (u, d)^T$ and their anti-particles u^c and d^c (i.e. those particles appear in three colors Red, Blue, Green). I have used left-handed particles and left handed anti-particles (ψ_L and ψ_L^c) to present the matter content. I could just as well have chosen left-handed particles and right-handed particles (since $\psi_R = C(\bar{\psi_R^c})^T$, where C is the charge conjugation operator in your favorite Dirac matrices representation). The left-handed leptons are $l = (\nu, e)^T$. I have added to the list a right-handed neutrino ν^c. This is usually left out of the Standard Model, but since we do observe a neutrino mass we can postulate a right-handed neutrino which will allow for a Dirac mass term for the neutrino (that's not the only way to give it a mass, but a very simple way). Finally, we have the Higgs sector, where $H = (h^+, h^0)^T$ (the superscripts indicate the electric charge). In the usual description of spontaneous symmetry breakdown (SSB), the Higgs potential is arranged so that the vacuum state of the field, transforms in a non-trivial way under $SU(2)$. The vacuum state is conventionally chosen such that

$$\langle H \rangle = (0, v)^T \tag{18}$$

This breaks the electroweak force to the familiar electromagnetic and weak forces $SU(2) \times U(1) \rightarrow U(1)_{EM}$. With this choice of vacuum (equation (18)) there are three broken generators, which will correspond to three Goldstone bosons. Those will be "eaten" by the gauge fields corresponding to the broken generators to give them mass. One generator is unbroken and therefore the associated gauge field will remain massless (the photon). The physical Higgs field will have a mass of the order of the SSB scale v (I'm not providing many details since those can be found in any treatment of the subject and reference [1] does an excellent job presenting all of this).

5 SU(5) Unification

$SU(N)$ is defined as the group of special (determinant is unity) unitary $N \times N$ matrices. The defining representation can be written as,

$$U(\vec{\theta}) = \exp(-i \sum_a \theta^a T^a) \tag{19}$$

We have $N^2 - 1$ generators which can be chosen to be hermitian and traceless. The conventional normalization for the generators is $Tr(T^a T^b) = \frac{1}{2}\delta_{ab}$. We will chose the following basis for our representation,

$$(T^a_b)_{cd} \equiv (T^a_b)^c_d \equiv \delta^c_b \delta^a_d - \frac{1}{N}\delta^a_b \delta^c_d \qquad a, b, c, d = 1...N \tag{20}$$

We wrote the row index as a superscript because it will allow us to distinguish between fields transforming according to the representation and those transforming according to the complex conjugate representation. The commutation relations of the generators can be easily worked out to be,

$$[T^a_b, T^{a'}_{b'}] = \delta^a_{b'} T^{a'}_b - \delta^{'a}_b T^a_{b'} \tag{21}$$

For a general representation of the Lie algebra just replace all the T^a_b's with t^a_b's and demand the same commutations relations of course (there is no content in the this sentence, just formality). Now, we will say that a set of fields ψ^c with $c = 1...N$, transform under the fundamental N dimensional representation if,

$$[t^a_b, \psi^c] = -(T^a_b)^c_d \psi^d \tag{22}$$

The hermitian conjugate $\chi_c = (\psi^c)^\dagger$ will transform under the complex conjugate representation $\bar{N}$ of the fundamental representation.

$$[t^a_b, \chi_c] = -(T^a_b(\bar{N}))^d_c \chi_d \tag{23}$$

where, recall that $T^a_b(\bar{N}) = -(T^a_b)^T = -T^b_a$. In general, all the irreducible representations of the Lie algebra can be obtained by symmetrizing and anti-symmetrizing direct products of fields in the N and $\bar{N}$ representations. In particular the $N^2 - 1$ adjoint representation fields φ^a_b (recall, in the adjoint representation we take our generators to be the structure constants, see equation (12)) transform as,

$$[t^a_b, \varphi^c_d] = -(T^a_b)^c_{c'} \varphi^{c'}_d - (T^a_b(\bar{N}))^{d'}_d \varphi^c_{d'} = -\delta^c_b \varphi^a_d + \delta^a_d \varphi^c_b \tag{24}$$

Ok, let'd cut the formalism and describe the Georgi-Glashow $SU(5)$ model. So as the name implies, we assume that our theory is an $SU(5)$ gauge theory. The Standard Model $SU(3) \times SU(2) \times U(1)$ is embedded as a subgroup in the following manner. The $SU(3)$ subgroup generators are,

$$T^\beta_\alpha - \frac{1}{3}\delta^\beta_\alpha T^\gamma_\gamma \qquad \alpha, \beta, \gamma = 1, 2, 3 \tag{25}$$

(There are 8 independent ones, as should be). The $SU(2)$ subgroup generators are,

$$T^r_s - \frac{1}{2}\delta^r_s T^t_t \qquad r, s, t = 4, 5 \tag{26}$$

(I will use (α, β, γ), (r, s, t) and (a, b, c) to denote $SU(3)$, $SU(2)$ and $SU(5)$ indices, respectively. The $U(1)$ hypercharge generators is,

$$Y = -\frac{1}{3}T^\alpha_\alpha + \frac{1}{2}T^r_r \tag{27}$$

Let's count them. 8 $SU(3)$ generators, 3 $SU(2)$ generators and 1 hypercharge to a total of 12 generators. So we have an additional 12 generators T^α_r and T^r_α (since $SU(5)$ has $5^2 - 1 = 24$ generators). Those generators

relate flavor to color and will therefore naturally lead to proton decay. However, we will of course devise some mechanism (e.g. Higgs) of breaking this symmetry (since we don't observe it in our energy scales), and those additional 12 generators will "eat" the 12 Goldstone bosons and become heavy. Very heavy. As heavy as our SSB scale, which as we shall see (or have seen already on our problem sets) is roughly $10^{16}GeV$. In our energy scales we can describe such a process the same way people described 4-fermion interactions in the Fermi theory of the weak force. Our interaction will be a non-renormalizable one, and suppressed by mass squared in the denominator. What mass? The mass of the heavy gauge mediators of course (in complete analogy with the Fermi theory and the heavy $W^{\pm}$, Z particles). Even though very small, it is detectable nonetheless, and constrains the unification energy.
The 24 generators are in the adjoint representation as always in gauge theories, and on restriction to the subgroup $SU(3) \times SU(2) \times U(1)$ they transform as,

$$\begin{array}{ccccccccccc} 24 = & (8,1,0) & + & (1,3,0) & + & (1,1,0) & + & (3,\bar{2},-\frac{5}{6}) & + & (\bar{3},2,+\frac{5}{6}) \\ & G^{\alpha}_{\beta} & & W^{\pm}, W^{0} & & B & & A^{\alpha}_{r} & & A^{r}_{\alpha} \end{array}$$

Where I have used (n_1, n_2, n_3) to indicate how those fields transform under $SU(3)$, $SU(2)$ and $U(1)$, respectively (e.g. (8,1,0) are the gluons which transform under the adjoint of $SU(3)$ but otherwise as singlets). Now for the fermions of the theory. I will specify how the left-handed particles transform. Each family of fermions (q,u,d,l,e, 15 in total) is put together in a $10 + \bar{5}$ (reducible) representation of $SU(5)$. Let's see how the field $\bar{5}$ transform under restriction to the $SU(3) \times SU(2) \times U(1)$ subgroup,

$$\bar{5} = (\bar{3}, 1, \frac{1}{3}) + (1, \bar{2}, -\frac{1}{2}) \tag{28}$$

(it is not so hard to verify this, and one way to do so is by the orthogonality relations for the group's characters). This looks familiar! $(\bar{3}, 1, \frac{1}{3})$ is how the d^c quark transforms and $(1, \bar{2}, -\frac{1}{2})$ is how the l doublet transform. nice. How does the 10 (which is an antisymmetric product of two 5's) reduce?

$$10 = (\bar{3}, 1, -\frac{2}{3}) + (3, 2, \frac{1}{6}) + (1, 1, 1) \tag{29}$$

mmmm... very familiar, u^c, q doublet and e^c fields. How nice. So we can identify,

$$\bar{5}: \quad \psi_{La} = \begin{pmatrix} d^c_1 \\ d^c_2 \\ d^c_3 \\ e^- \\ -\nu_e \end{pmatrix}_L \tag{30}$$

$$10: \quad \psi^{ab}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u^c_3 & -u^c_2 & -u^1 & -d^1 \\ -u^c_3 & 0 & u^c_1 & -u^2 & -d^2 \\ u^c_2 & -u^c_1 & 0 & -u^3 & -d^3 \\ u^1 & u^3 & u^3 & 0 & -e^+ \\ d^1 & d^2 & d^3 & e^+ & 0 \end{pmatrix} \tag{31}$$

The corresponding kinetic term in the Lagrangian is given by,

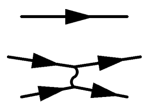
$$\mathcal{L}_f = i\bar{\psi_L}_a \sigma^{\mu} (D_{\mu}\psi_L)^a + i\bar{\psi_L}^{ab} \sigma^{\mu} (D_{\mu}\psi_L)^{ab} \tag{32}$$

$$= \bar{\psi_L}_a \sigma^{\mu} \left(i\partial_{\mu}\delta^a_b + \frac{g_5}{\sqrt{2}} (A_{\mu})^a_b \right) \psi^a_L + \bar{\psi_L}^{ac} \sigma^{\mu} \left(i\partial_{\mu}\delta^a_b + \frac{2g_5}{\sqrt{2}} (A_{\mu})^a_b \right) \psi_L)^{bc} \tag{33}$$

We can write it now in terms of the $SU(3) \times SU(2) \times U(1)$ fields, but this is a real mess. The important thing is the fact that what we used to call different gauge couplings (g,g' and g_s) are now all related to the $SU(5)$ gauge coupling g_5 (note however that $g' = \sqrt{\frac{3}{5}} g_5$). Expanding the expression also shows us that we will have Baryon and lepton non-conserving terms. Let's write one for example (you can find the whole expression on page 263 of reference [1]),

$$\frac{g_5}{\sqrt{2}} (A_4^\alpha)_\mu (\bar{d}_{L\alpha} \sigma^\mu e_L^+) \tag{34}$$

Assuming that $SU(5)$ breaks down at some very high energy M_{GUT}, much higher than the mass of the proton, we will have to integrate out the heavy gauge fields which correspond to the broken generators. This will lead to (through terms as the above) a 4-fermion interaction such as shown below,



It is easy to estimate the life-time of the proton. Since the mediator has mass of say M_{GUT} the amplitude will have this mass squared in the denominator. We also get a factor of $\alpha_5 = g_5/4\pi$ in the numerator. The life-time is the inverse of the square of the amplitude, so in order to have the units work out we need the proton mass m_p, to get,

$$\tau_p \sim \frac{1}{\alpha_5^2} \frac{M_{GUT}^4}{m_p^5} \tag{35}$$

This implies $M_{GUT} \sim 10^{16} GeV$.

It is not at all obvious that this model is anomaly free. But it is. People have classified all the anomaly free groups and it turns out (this is story telling, I'm not proving anything) that of all the simple Lie groups only the $SU(N)$'s for $N > 2$ have anomalies. In particular $SO(10)$ is anomaly free. This is important because the reducible $10 + \bar{5}$ representation we have chosen for our fermion content can be embedded in a 16 spinor representation of $SO(10)$. Since SO(10) is anomaly free this specific $SU(5)$ model is anomaly free.
There are many other things to discuss concerning this model, like SSB mechanism, mass generation for the fermions and mixing matrices (generalized KMS matrices), charge quantization (now that the leptons and quarks lay in the same representation, it is clear why the charge of the electron and proton are exactly opposite) and baryon anti-baryon asymmetry. But I will not include those issues in these notes. Let me just summarize some key points (there are many more aspects, but...) and I'll try to give a better summary in the lecture,

1. $SU(5)$ incorporates $SU(3) \times SU(2) \times U(1)$ as a maximal subgroup.

2. Electric charge is quantized. Since the electric charge is now an $SU(5)$ operator we know that its trace must be in every representation and therefore the charges of the different fields sitting in a representation must be related. In particular, take the $\bar{5}$ representation from which we see that $3Q_{d^c} + Q_{e^-} = 0$.

3. It's anomaly free.

6 References

1. P. Langacker, Phys. Rep. 72, 185 (1981). I found this to be a very comprehensive reference. He doesn't go into all the little details, but rather provide a very broad review of the subject with appropriate references.

2. J. P. Elliott and P. G. Dawber, Symmetry in Physics, Oxford University Press, 1979. I found this book extremely clear in presenting group theory. Full of physical examples, and the theory is developed in a very consistent manner. The only drawback is that it doesn't cover some modern topics like unification, supersymmetry etc. But overall an excellent textbook.

3. H. Georgi, Lie Algebras in Particle Physics, Advanced Book Program, Westview Press, 1999. The choice of topics is aimed at particle physics students, which is the strength of the book. However, I found the presentation to be lacking in many places. Many details are skipped and claims are made without much backup. Not that it is a bad book, but it forces the reader to really work through everything and verify stuff.

4. S. Weinberg, The Quantum Theory of Fields - Modern Applications, Cambridge University Press, 1996. Even though the notation is horrendous, I find Weinberg very insightful. He presents things in a very consistent way, and the reader is left with little doubt. But the book is very dense and takes a long while to digest.