Deconstructing CCWZ

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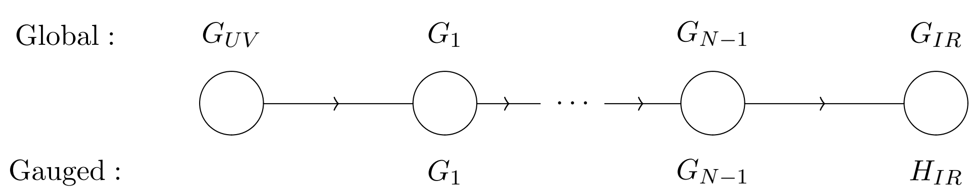
1 Motivation

We are very familiar with moose diagrams as a convenient model building tool, but at first glance, they seem to be used in two very different contexts. First, moose diagrams can summarize certain symmetry breaking patterns, such as chiral symmetry breaking in QCD [1]. Second, moose diagrams can be used deconstruct an extra dimension [2]. In the case of deconstruction, *any* non-gravitational fifth dimension can be approximated by a linear moose with an arbitrary number of sites, and if we ignore back-reaction, the effect of warped geometry [3] can be mimicked by the choice of decay constants for the various link fields. However, the application of mooses to symmetry breaking patterns appears to be less universal. Indeed, there is no obvious way to construct a moose to describe a generic G/H non-linear sigma model.

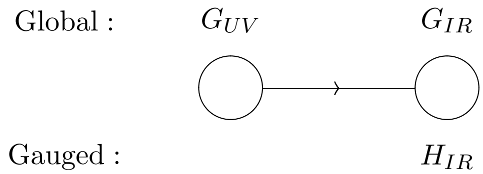
In this note, we will show how to understand *any* G/H symmetry breaking pattern in terms of a very simple two-site moose. Our inspiration is the AdS/CFT correspondence and its phenomenological interpretation [4]. In order to realize the desired G/H non-linear sigma model, we imagine a 4D quasi-CFT with a global G symmetry that is spontaneously broken to H by QCD-like confinement. The Goldstones of G/H arise analogously to the pions of QCD. The AdS dual of this picture is a slice of AdS$_5$ space with a bulk G gauge symmetry that is reduced to H on the IR brane and completely broken on the UV brane.

Now, we can imagine deconstructing the fifth dimension of this AdS model. The link fields in the moose are precisely the Wilson lines constructed out of A_5.



(1)

Going the extreme where we only introduce sites corresponding the UV and IR branes, we arrive at a moose diagram that will be the starting point for our analysis:



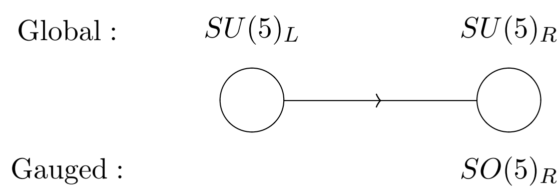
(2)

At low energies, this is supposed to describe a G/H non-linear sigma model involving only global symmetries, but in this AdS/CFT inspired moose, the subgroup H has become a gauge symmetry!

To understand why this is plausible, note that the counting of the degrees of freedom works out properly. The link field contains a Gs worth of Goldstones, and an Hs worth are eaten by the H_{IR} gauge group. If we integrate out the heavy gauge fields, we are left with a G/Hs worth of Goldstones, as expected. Of course, we have to contend with the effect of these heavy gauge fields, but we will see that they play a rather benign role. In the low energy theory, we can simply do naive dimension analysis with a cutoff $\Lambda = gf$, where g is the gauge coupling of H_{IR} and f is the Goldstone decay constant for the link field. In the limit of strong coupling $g \to 4\pi$, we produce a G/H non-linear sigma model with standard NDA rules.

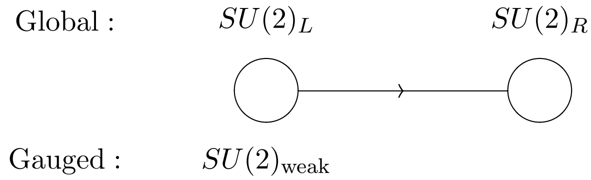
As we will see, the moose in equation (2) has a beautiful translation into the original G/H language of CCWZ [5, 6]. The link field in the moose diagram *is* the field that non-linearly realizes the G symmetry. Also, some aspects of CCWZ which might seem obscure or formal in the CCWZ language become necessary consequences of the H_{IR} gauge symmetry. Of course, the low energy phenomenology of the moose in equation (2) is identical to CCWZ G/H phenomenology. In the limit $g \to 4\pi$, it is not even meaningful to talk about the heavy gauge bosons because their longitudinal modes are so strongly coupled that the theory needs a UV completion anyway. In other words, the G/H moose is not a UV completion of a G/H non-linear sigma model but rather an alternate description of it.

Then again, the G/H moose could inspire different kinds of UV completions. In particular, the G/H moose offers a new way to realize the $SU(5)/SO(5)$ Littlest Higgs [7]. In the original conception of the Littlest Higgs, the imagined UV completion was an $SO(N)$ confining theory with 5 Weyl fermions Ψ_i transforming in the fundamental of $SO(N)$. The $SU(5)$ global symmetry of the fermions would then be broken to $SO(5)$ by a $\langle \Psi_i \Psi_j \rangle$ condensate. But the CCWZ moose for $SU(5)/SO(5)$ suggests a different kind of UV completion:



(3)

This moose practically screams out to be interpreted as an $SU(N_c)$ confining theory with $N_f = 5$ that exhibits chiral symmetry breaking. The pions of this model fill out an $SU(5)$, but an $SO(5)$s worth are eaten by the $SO(5)_R$ subgroup of $SU(5)_R$, leaving the desired $SU(5)/SO(5)$ non-linear sigma model. This moose is identical in spirit to technicolor with hypercharge turned off.



(4)

In the case of technicolor, $SU(2)_{\text{weak}}$ eats all of the $SU(2)$ pions, but there is no conceptual problem imagining a technicolor model with more quark flavors. Indeed, the phenomenologically dangerous light Goldstones in $N_f > 2$ technicolor become the desired light Goldstones in the $SU(5)/SO(5)$ moose.

2 Review of CCWZ

Before analyzing the G/H moose, we review the basics of the CCWZ formalism. The CCWZ prescription is a generic way to parametrize the Goldstone bosons π^a arising from a G/H symmetry breaking pattern. If T^a are the generators of H, and X^a are the generators of G/H, we can introduce a field

$$\xi = e^{i\pi^a X^a/f} \tag{5}$$

that transforms as

$$\xi \to g\xi U^\dagger, \tag{6}$$

where g is an element of G, and $U \in H$ is a function of the π^a and furnishes a non-linear representation of G. For general G and H, however, the Goldstone kinetic terms are not particularly simple. We can decompose

$$\xi^\dagger \partial_\mu \xi \equiv v_\mu^a T^a + p_\mu^a X^a, \tag{7}$$

and the objects $v_\mu = v_\mu^a T^a$ and $p_\mu = p_\mu^a X^a$ transform as

$$v_\mu \quad \to \quad U(v_\mu + \partial_\mu)U^\dagger, \tag{8}$$
$$p_\mu \quad \to \quad U p_\mu U^\dagger. \tag{9}$$

The field v_μ transforms like a connection, and we can use p_μ to write down a Goldstone kinetic term

$$\mathcal{L}_{\text{kinetic}} = f^2 \operatorname{tr} p^\mu p_\mu^\dagger, \tag{10}$$

but the form of p_μ depends heavily on the specific groups G and H.

However, if the Lie algebra G/H is a so-called symmetric space, we can simplify the Goldstone kinetic terms. The Lie algebra of a symmetric space satisfies the schematic commutation relations

$$[T, T] \sim T, \qquad [T, X] \sim X, \qquad [X, X] \sim T. \tag{11}$$

Therefore, a symmetric space has an automorphism

$$T^a \to T^a, \qquad X^a \to -X^a. \tag{12}$$

By applying this automorphism to equation (7), we find that the object p_μ is simply given by

$$p_\mu = \frac{1}{2}\left(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger\right), \tag{13}$$

and we can rewrite the Goldstone kinetic term in equation (10) as

$$\mathcal{L}_{\text{kinetic}} = \frac{f^2}{4} \operatorname{tr} |\partial_\mu \Sigma|^2. \tag{14}$$

We have defined

$$\Sigma = \xi \tilde{\xi}^\dagger = \xi^2 = e^{2i\pi^a X^a / f}, \tag{15}$$

and $\tilde{\xi}$ is the image of ξ under the automorphism. Looking at equation (6), we see that $\Sigma = \xi\tilde{\xi}^\dagger$ transforms as

$$\Sigma \to g \Sigma \tilde{g}^\dagger, \tag{16}$$

where $\tilde{g}$ is the image of g under the automorphism. Therefore, in symmetric spaces we can construct a the Goldstone matrix Σ that is an element of G/H but it transforms linearly under G.

The classic example of a theory with a $X^a \to -X^a$ automorphism is chiral symmetry breaking in QCD:

$$SU(N_f)_L \times SU(N_f)_R / SU(N_f)_D. \tag{17}$$

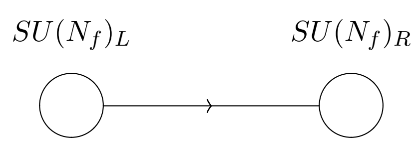
The automorphism simply exchanges the left and right groups. In that case, the associated Σ field has transformation properties

$$\Sigma \to g_L g_R \Sigma (g_R g_L)^\dagger, \tag{18}$$

where $g_i \in SU(N_f)_i$. (Note that g_L and g_R commute.) If we consider the transformation under $SU(N_f)_L$ alone, we find that (recall that $\tilde{g}_L = g_R$)

$$\Sigma \to g_L \Sigma g_R^\dagger. \tag{19}$$

Therefore, we can describe chiral symmetry breaking in terms of the following moose diagram:



(20)

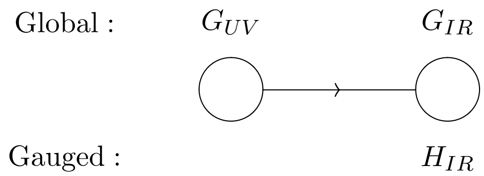
For other symmetric spaces, however, there is no obvious moose description. An example is the $SU(5)/SO(5)$ Littlest Higgs theory that can be described in terms of a Σ field that transforms as

$$\Sigma \to V \Sigma V^T, \tag{21}$$

where $V \in SU(5)$. (In this case, the automorphism is related to complex conjugation.) Because there is no moose diagram for the Littlest Higgs, it is hard to see how the Littlest Higgs could arise from chiral symmetry breaking in a QCD-like $SU(N_c)$ theory. However, as we have already argued, the moose in equation (3) describes a $SU(5)/SO(5)$ non-linear sigma model at low energies, suggesting that the Littlest Higgs could come from chiral symmetry breakings in a QCD-like theory with $N_f = 5$.

3 Understanding the G/H Moose

We now return to the proposed moose in equation (2) that is supposed to yield a G/H non-linear sigma model at low energies:



With malice of forethought, we will call the link field between the UV and IR sites ξ. Under $G_{UV} \times G_{IR}$, the link field transforms as

$$\xi \to g_{UV} \xi g_{IR}^{\dagger}. \tag{22}$$

Because the subgroup H_{IR} of G_{IR} has been gauged, though, the surviving global symmetry on the IR site is just H_{IR} (assuming that there is no subgroup of G/H that commutes with H). Therefore, under the surviving global symmetry

$$\xi \to g_{UV} \xi h_{IR}^{\dagger}, \tag{23}$$

which is precisely the transformation law in equation (6)! Apparently, h_{IR} furnishes a non-linear representation of G_{UV}. Note that after H_{IR} eats the relevant Goldstones, ξ is just an element of G/H, so in a symmetric space, we can construct the non-linear sigma field

$$\Sigma = \xi^2 \to g_{UV} \Sigma \tilde{g}_{UV}^{\dagger}. \tag{24}$$

What about the heavy gauge bosons? One might worry that they could mix with ξ, and the resulting low energy theory wouldn't just be a G/H non-linear sigma model. However, we will see that they mix in *precisely* the right way to realize the mapping between the moose in equation (2) and the CCWZ Goldstone kinetic equation (10). In fact, the heavy gauge bosons will be mapped to the connection field v_μ in equation (8), such that fields charged only under H_{IR} can be consistently coupled to fields charged under G_{UV}.

After going to a unitary gauge where the gauge bosons in H_{IR} eat the relevant Goldstones, we are left with a link field $\xi = e^{i\pi^a X^a/f}$ with the following leading order Lagrangian

$$\mathcal{L}_{\text{kinetic}} = f^2 \operatorname{tr} |D_\mu \xi|^2, \qquad D_\mu \xi = \partial_\mu \xi - ig\xi A_\mu, \tag{25}$$

where $A_\mu = A_\mu^a T^a$. Expanding this Lagrangian

$$\mathcal{L}_{\text{kinetic}} = f^2 \operatorname{tr} |\partial_\mu \xi|^2 + \frac{g^2 f^2}{2} A_\mu^a A^{\mu a} + 2igf^2 A_\mu^a \operatorname{tr}\left(T^a \xi^\dagger \partial_\mu \xi\right). \tag{26}$$

We recognize the combination $\xi^\dagger \partial_\mu \xi$ from equation (7), and we find that the equation of motion for A_μ^a sets

$$igA_\mu = v_\mu, \tag{27}$$

where we have used the orthogonality relation $\operatorname{tr} T^a T^b = \delta^{ab}/2$. Therefore, we see that the heavy gauge fields play the same role as the connection v_μ in the CCWZ formalism.

What about the ξ kinetic terms? Integrating out the A_μ^a fields from equation (26) to leading order in derivatives:

$$\mathcal{L}_{\text{kinetic}} = f^2 \,\text{tr}\, |\xi^\dagger \partial_\mu \xi|^2 + 2f^2 \sum_a \left(\text{tr}\, T^a \xi^\dagger \partial_\mu \xi \right)^2 . \tag{28}$$

Expressed in terms of p_μ and v_μ (note that $v_\mu^\dagger = -v_\mu$):

$$\mathcal{L}_{\text{kinetic}} = f^2 (\text{tr}\, p^\mu p_\mu^\dagger + \text{tr}\, v^\mu v_\mu^\dagger) + \frac{f^2}{2} \sum_a v_\mu^a v^{\mu a} = f^2 \,\text{tr}\, p^\mu p_\mu^\dagger, \tag{29}$$

which is precisely equation (10)! So the moose in equation (2) does indeed yield the the entire CCWZ formalism for a G/H symmetry breaking pattern. If we want to introduce fields with well defined transformation laws under H, we simply attach them to the IR site and the A_μ gauge fields will dutifully play the role of the v_μ connection. If we want to gauge a subgroup of G, we simply gauge that subgroup on the UV site, precisely what we expect from the AdS/CFT correspondence.

What about higher derivative interactions induced by integrating out A_μ? For example, the next-to-leading operator coming from equation (26) is

$$\mathcal{L}_{\text{NLO}} \sim \frac{f^2}{2\Lambda^2} \,\text{tr}\, (V_{\mu\nu} V^{\mu\nu}) , \tag{30}$$

where $\Lambda = gf$ is the mass of the heavy gauge field, and

$$V_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + [v_\mu, v_\nu]. \tag{31}$$

We know that v_μ must appear in $\mathcal{L}_{\text{NLO}}$ in the combination $V_{\mu\nu}$ because v_μ transforms like a connection (see equation (8), but replace U with an H_{IR} gauge transformation). To get a better feel for $V_{\mu\nu}$, we can go to a symmetric space where

$$v_\mu = \frac{1}{2} \left(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger \right) . \tag{32}$$

It is straight forward to show that

$$V_{\mu\nu} = p_\mu p_\nu^\dagger - p_\nu p_\mu^\dagger = \xi \left(\partial_\mu \hat{\Sigma} \partial_\nu \hat{\Sigma}^\dagger - \partial_\nu \hat{\Sigma} \partial_\mu \hat{\Sigma}^\dagger \right) \xi^\dagger, \tag{33}$$

where $\hat{\Sigma} = \xi^2/2$ is normalized to give the right NDA counting. This leads to

$$\mathcal{L}_{\text{NLO}} \sim \frac{1}{g^2} \,\text{tr}\, \left(\partial_\mu \hat{\Sigma} \partial_\nu \hat{\Sigma}^\dagger \right)^2 - \frac{1}{g^2} \,\text{tr}\, \left(\partial_\mu \hat{\Sigma} \partial^\mu \hat{\Sigma}^\dagger \right)^2 . \tag{34}$$

The coefficients $1/g^2$ are precisely the NDA estimate for these operators with g replacing 4π.

If we had fermions ψ that transformed under H_{IR}, they would appear in the original Lagrangian as

$$\mathcal{L}_{\text{fermions}} = \sum_\psi \bar{\psi} \bar{\sigma}^\mu D_\mu \psi, \qquad D_\mu = \partial_\mu + i g A_\mu^a T_\psi^a, \tag{35}$$

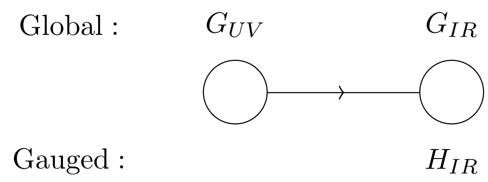
where T_ψ^a are the generators of H_{IR} in the ψ representation. When we integrate out A_μ, to leading order we can simply set $igA_\mu^a = v_\mu^a$, but to next-to-leading order, we generate a four fermion operator

$$\mathcal{L}_{\text{four fermion}} = \frac{g^2}{\Lambda^2} J_\mu^a J^{\mu a}, \qquad J_\mu^a = \sum_\psi \bar{\psi} \bar{\sigma}^\mu T_\psi^a \psi. \tag{36}$$

The coefficient of this operator $g^2/\Lambda^2 = 1/f^2$ is indeed consistent with NDA.

4 Summary

We have seen that any G/H symmetry breaking pattern can be realized by a simple two-site moose:



This construction was motivated by the AdS/CFT correspondence, but it should be emphasized that this moose could arise from a theory that looks nothing like a quasi-conformal field theory.

References

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