Model of Black Hole Formation in $2+1$ Dimensions

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Overview

Goal: To determine whether Black Hole formation is a tunneling process.

$$\int^{R_0} P dR = \infty ?$$

Current literature examining the problem using other methods suggests that it is.

Model: A head-on collision of two massless particles in $2+1$ dimensional Gravity with negative cosmological constant.

Opportunity: To understand Gravity as a simple algebraic/geometric phenomena.

Caveat: $2+1 \neq 3+1$.

Outline

1. Compare: Particle in Black Hole Background
2. Model: Particle Collision in $2+1$ Dimensions
   (a) Coordinates in AdS Space
   (b) Single Particle in AdS Space
   (c) Two Particle Collision in AdS Space
3. Calculations: From AdS to BTZ
   (a) Moving to the "Center of Mass" Frame
   (b) Moving to BTZ Black Hole Coordinates
   (c) Approximating a Relative Hamiltonian near the Black Hole Horizon
   (d) Calculating the Tunneling Quantity
4. Looking Forward

Semiclassical Amplitudes

In Feynman Path Integral approach to Quantum Mechanics, amplitudes related to action.

$$A = \sum_{\text{all paths}} e^{\frac{i}{\hbar} S}$$

Semiclassical approximation: only consider classical path.

$$A \approx e^{\frac{i}{\hbar} S_{classical}}$$

Probability is equal to $|A|^2$, so for finite real action, probability is 1.

But for infinite action, we can perform $i\epsilon$ deformation to recover an imaginary component.

Thus, if the action for a process is infinite, there is a possibility for semiclassical suppression or enhancement.

The Action and $\overline{S}$

The Lagrangian is given by $L = P\dot{R} - H$ where H is the Hamiltonian. This yields the action:

$$S_{classical} = \int (P\dot{R} - H)dt = \int PdR - ET.$$

We are looking for possible divergences in the action. Clearly $-ET$ diverges as $T \to \infty$, but this is true for any energy conserving process, so it is not particularly interesting.

What if $P \to \infty$ at some $R = \mu$?

Thus, we will be interested in the following quantity:

$$\overline{S} = \int^{\mu} PdR$$

By calculating $\overline{S}$ for our black hole model, we can figure out whether or not Black Hole Formation is suppressed/enhanced or not.

Falling into a Black Hole

Models of Black Hole Formation based on a test particle falling in a Black Hole background suggest exponential suppression.

Problem: This model ignores kinetic energy of test particle, so total energy of system is skewed!

Result: Different momentum divergence relations and possible miscalculation of $\overline{S}$.

Let's see where the exponential suppression comes from by examining the action for a particle falling in a BTZ Black Hole. (This is the same black hole we will create in our model.)

Falling into a Black Hole con't

BTZ Black Hole Metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}$$

$$f(r) = r^2 - r_0^2$$

The mass of the black hole is related by $M = r_0^2$.

The associated Lagrangian for test particle:

$$L = -m\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}}$$

The momentum conjugate to r:

$$p = \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{f(r)\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}}}$$

$$\dot{r}^2 = \frac{f(r)^3 p^2}{m^2 + f(r)p^2}$$

Falling into a Black Hole con't

For simplicity consider only massless test particles. Limit as $m \to 0$:

$$L \to 0 \text{ and } \dot{r} \to \pm f(r).$$

Note that near the horizon, $f(r) \to 0$, therefore $\dot{r} \to 0$. In particular:

$$\dot{r} \propto M - r^2$$

(We will see this again in our black hole model.)

Hamiltonian is given by $H = p\dot{r} - L$. For $m = 0$:

$$H = f(r)|p|$$

$$p = \frac{H}{r_0^2 - r^2} \approx \frac{H}{2r_0(r_0 - r)}$$

Momentum diverges *linearly* as the particle reaches the horizon. This linear divergence of the momentum will yield an infinity in $\overline{S}$.

Falling into a Black Hole con't

$$\overline{S} = \int^{r_0} pdr \; = \; E\int^{r_0} \frac{dr}{r_0^2 - r^2}$$
$$= \frac{E}{r_0} \operatorname{arctanh} 1 = \infty$$

By $i\epsilon$ deformation, we can extract an imaginary component to the action.

$$A = e^{-\beta}$$

(If we were talking about Hawking radition, β would be related to Hawking temperature of black hole.)

Thus, any model of black hole formation based on a test particle falling in a black hole background will suggest exponential suppression.

But our model has a softer *logarithmic* divergence, so $\overline{S}$ will be finite.

Why Our Model is Better

Simple: Collision of Two Photons in $2+1$ Dimensions

Exactly Solvable: No approximations in relative motion equation. (There is an approximation in calculation of Hamiltonian, but we've checked that this does not change main result.)

Intuitive: Why should black hole formation be suppressed? We expect black hole formation to be a geometric process. Once particles have entered effective black hole horizon, we should get a black hole, right?

But...

AdS Space

Vacuum Solution to Einstein Equation with Negative Cosmological Constant:

$$x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 = \ell^2$$

Description in terms of $SL(2)$ group (real 2×2 matrices with determinant 1):

$$\gamma_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

AdS Coordinates:

$$z = x_{-1}\gamma_{-1} + x_0\gamma_0 + x_1\gamma_1 + x_2\gamma_2.$$

$$\det z = 1 \implies x_i x^i = 1$$

$$g_{ij} = \operatorname{diag}(1, 1, -1, -1)$$

AdS Space con't

Cylindrical Coordinates (χ, ϕ, t):

$$\begin{aligned} x_{-1} &= \cosh\chi\cos t \\ x_0 &= \cosh\chi\sin t \\ x_1 &= \sinh\chi\cos\phi \\ x_2 &= \sinh\chi\sin\phi \end{aligned}$$

Line element:

$$ds^2 = -\cosh^2\chi\, dt^2 + d\chi^2 + \sinh^2\chi\, d\phi^2$$

$SL(2)$ group element:

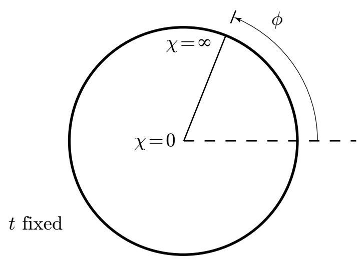
$$\begin{aligned} z &= \cosh\chi(\cos t\,\gamma_{-1} + \sin t\,\gamma_0) \\ &\quad + \sinh\chi(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) \\ &= e^{\frac{1}{2}(t+\phi)\gamma_0} e^{\chi\gamma_1} e^{\frac{1}{2}(t-\phi)\gamma_0} \end{aligned}$$

AdS Space con't

For graphing purposes:

$$r = \tanh\frac{\chi}{2}$$

Poincaré Disc: AdS space mapped onto circular time slices of radius 1.

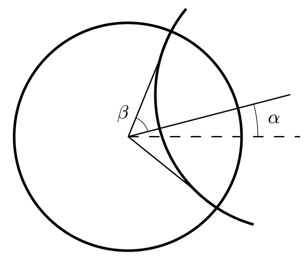


Note that because of the metric, proper time runs at different speeds depending on χ. In particular, at $\chi = \infty$, $d\tau = \infty$ even if dt is finite.

AdS Space con't

Geodesics (Wedge Edges):

$$\tanh\chi\cos(\phi - \alpha) = \cos\beta$$



Geodesics (Particle Trajectories):

$$\tanh\chi = \tanh\xi\sin t, \quad \phi = \theta.$$

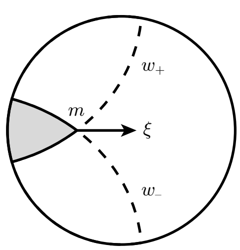
ξ is a *rapidity.*

$$v = \tanh\xi,$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\xi, \qquad \frac{v}{\sqrt{1-v^2}} = \sinh\xi.$$

Particle in AdS Space

In $2+1$ dimensional Gravity, the introduction of a particle cuts out a space-time wedge.



The wedge is defined by a *holonomy.* The wedge edges are geodesics in AdS space. The particle trajectory is given by the intersection of the wedge edges.

(This follows from applying the Einstein equation. The details are unimportant.)

The Holonomy

The only information we have about the particle is the momentum 3-vector.

$$\begin{aligned} p^0 &= \text{energy} = \gamma m = m\cosh\xi \\ p^1 &= x\text{-momentum} = \gamma m v_x = m\sinh\xi\cos\theta \\ p^2 &= y\text{-momentum} = \gamma m v_y = m\sinh\xi\sin\theta \end{aligned}$$

The holonomy is a packaging of the momentum 3-vector in the $SL(2)$ group.

$$u = e^{p^a\gamma_a} = \cos m\,\mathbf{1} + \frac{\sin m}{m}p^a\gamma_a$$

Massless particle ($\theta = 0, E = \tan\epsilon$):

$$\vec{p} = (\tan\epsilon, \pm\tan\epsilon, 0)$$

$$u = \mathbf{1} + \tan\epsilon(\gamma_0 \pm \gamma_1)$$

The Wedge Edge

Assume symmetry $\phi \to -\phi$. Wedge edges:

$$\begin{aligned} w_\pm &= \cosh\chi(\cos t\,\gamma_{-1} + \sin t\,\gamma_0) \\ &\quad + \sinh\chi(\cos(\pm\phi)\,\gamma_1 + \sin(\pm\phi)\,\gamma_2) \end{aligned}$$

Holonomy must send w_- to w_+:

$$u^{-1} w_- u = w_+$$

$$w_\pm : \tanh\chi \sin(\epsilon \pm \phi) = \sin t \sin\epsilon$$

Compare to circular geodesic:

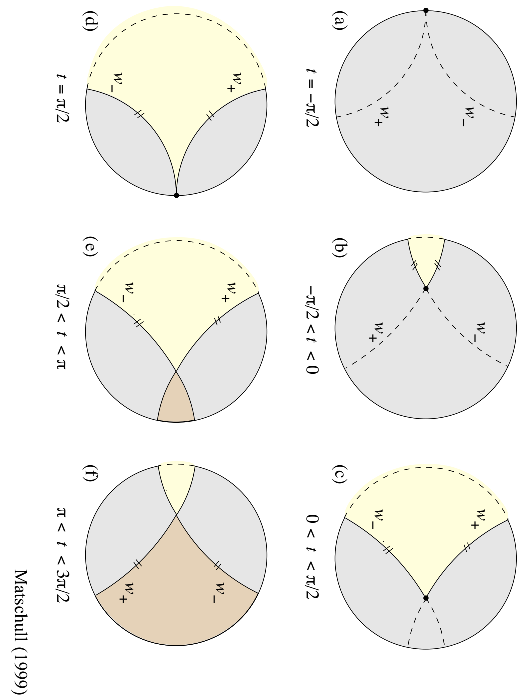
$$\tanh\chi\cos(\phi - \alpha) = \cos\beta$$

$$\pm\alpha = \frac{\pi}{2} - \epsilon$$

$$\cos\beta = \sin t \sin\epsilon$$

Particle trajectory:

$$\tanh\chi = \sin t$$

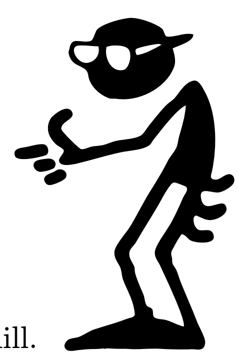


Gravity Minus One

Particles (massive or massless) introduce topological changes, but curvature away from the particle remains unchanged.

Topological change defined by holonomy, which in turn is defined by particle momentum 3-vector.

Algebraic construct ($SL(2)$) defines geometric gravity (AdS minus wedge).



Chill.

Collision in AdS Space

Particle 1: $\vec{p} = (\tan\epsilon, \tan\epsilon, 0)$

Particle 2: $\vec{p} = (\tan\epsilon, -\tan\epsilon, 0)$

Physics of the collision depends on the value of ϵ (Black Hole vs. Mass Lump)

When particles are close to each, the effective holonomy is:

$$\begin{aligned} u_{\mathit{eff}} &= u_1 u_2 \text{ or } u_2 u_1 \\ &= (1 - 2\tan^2\epsilon)\mathbf{1} + 2\tan\epsilon(\gamma_0 \pm \tan\epsilon\,\gamma_2) \end{aligned}$$

To find effective mass/momentum:

$$u_{\mathit{eff}} = \cos M\,\mathbf{1} + \frac{\sin M}{M} P^a \gamma_a$$

$$\sin\frac{M}{2} = \tan\epsilon$$

$$v_x = 0, v_y = \pm\tan\epsilon$$

Black Hole or Mass Lump?

Mass Equation:

$$\sin\frac{M}{2} = \tan\epsilon$$

Mass Lump: $0 < \epsilon < \frac{\pi}{4}$

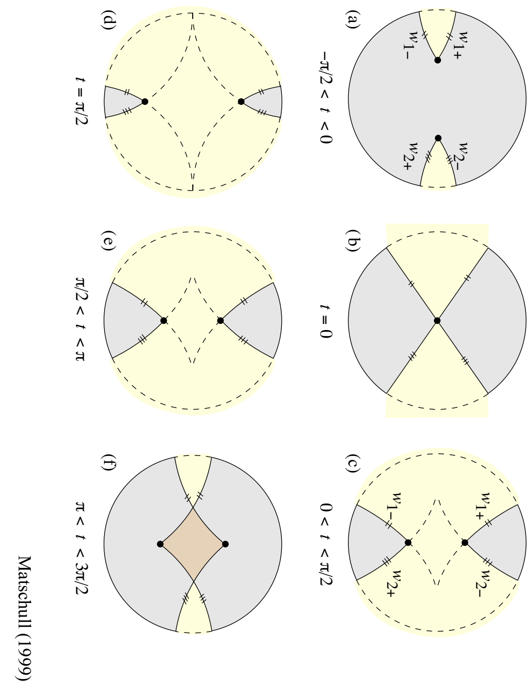
Black Hole (imaginary M): $\frac{\pi}{4} < \epsilon < \frac{\pi}{2}$

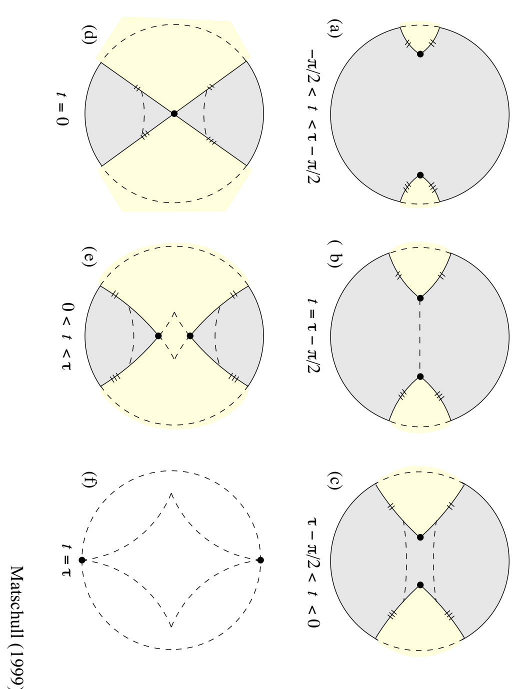
Velocity Equation:

$$v_y = \pm\tan\epsilon$$

$0 < \epsilon < \frac{\pi}{4}$: Ordinary Particle

$\frac{\pi}{4} < \epsilon < \frac{\pi}{2}$: Tachyon?





The Horizon

When particles join to form the "tachyon" they travel on space-like geodesics:

$$\tanh\chi = \sin t\tan\epsilon, \phi = \frac{\pi}{2}$$

Space-time ends when particle reaches $\tanh\chi = 1$:

$$\sin\tau = \frac{1}{\tan\epsilon}$$

Trace a backward light cone from $\tanh\chi = 1, t = \tau, \phi = \pi/2$:

$$\tanh\chi\cos\left(\phi\pm\frac{\pi}{2}\right) = \cos(\tau - t)$$

Looking at the dashed line on the previous slide, "tachyon" is shielded by black hole horizon.

The Horizon con't

Using the metric, it is possible to measure horizon length, 2μ. It is a constant!

$$\cosh\frac{\mu}{2} = \tan\epsilon$$

Our Black Hole really is a black hole!

1. Conserved horizon length
2. Space-time ends on singularity
3. Cosmic censorship of tachyons

But...

We want to move into a coordinate system where black hole properties are move evident.

The "Center of Mass" Frame

Does joint particle really have two velocities? We want to eliminate this "coordinate" effect.

Can we move to a coordinate system where:

$$u_{\mathit{eff}} = e^{M\gamma_0}?$$

Not possible for $\epsilon > \frac{\pi}{4}$. What is possible is:

$$u_{\mathit{eff}} = e^{-\mu\gamma_1}$$

In this way, μ is an analytic extension of M.

Lorentz transforming u_1 and u_2 to find μ is tedious. Instead, we can start with μ and find particle momenta that result in the proper effective holonomy.

The "Center of Mass" Frame con't

Particle 1: $\vec{p} = (\tan\epsilon_1, -\tan\epsilon_1\cos\theta, -\tan\epsilon_1\sin\theta)$

Particle 2: $\vec{p} = (\tan\epsilon_2, -\tan\epsilon_2, 0)$

Looking for a solution to:

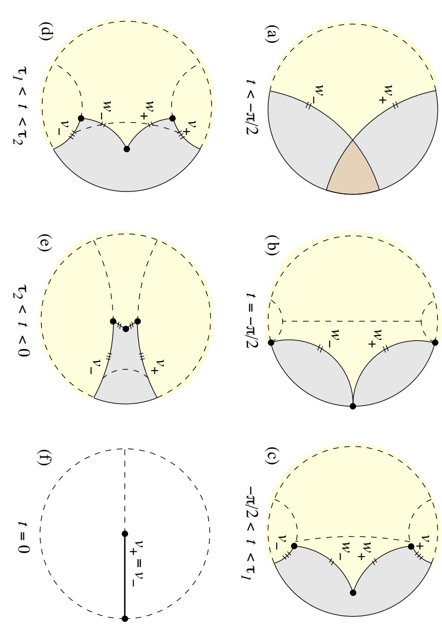
$$u_1 u_2 = e^{-\mu\gamma_1}$$

In terms of μ:

$$\tan\epsilon_1 = \coth\frac{\mu}{2}\cosh\mu$$

$$\sin\theta = \tanh\mu$$

$$\tan\epsilon_2 = \coth\frac{\mu}{2}$$



Matschull (1999)

Geodesic Distance

Path connecting particles given by:

$$w_+ : \tanh\chi \sin(\epsilon_2 + \phi) = -\sin t \sin\epsilon_2$$

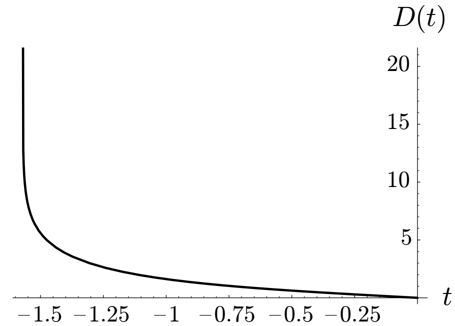
We can use the metric to find distance from particle 1 ($\phi = \theta$) to particle 2 ($\phi = \theta$).

$$D(t) = -2 \ \text{arctanh}\left(\frac{\sin t}{\sqrt{1 + \cos^2 t \coth\mu/2}}\right)$$

Does this function exhibit any unusual features as particles pass horizon?

Geodesic Distance con't

Graph for $\coth\mu/2 = 2$:



Thaler (2002)

Note smooth transition through horizon and to singularity.

Moving to BTZ Coordinates

BTZ Black Hole:

A solution to the Einstein equation similar in spirit to the Schwarzschild Black Hole. Closely related to AdS space.

$$ds^2 = -\left(\frac{\tilde{r}^2}{\ell^2} - 8GM\right)d\tilde{t}^2 + \frac{d\tilde{r}^2}{\frac{\tilde{r}^2}{\ell^2} - 8GM} + \tilde{r}^2 d\tilde{\phi}^2$$

Horizon: $\mu = \ell\sqrt{8GM}$

$$T = \mu\left(\tilde{t} + \frac{\pi}{2}\right) \qquad \frac{\mu\pi}{2} < T < \infty$$

$$R = \frac{\tilde{r}}{\mu} \qquad 1 < R < \infty$$

$$\Phi = \mu\tilde{\phi} \qquad -\mu < \Phi < \mu$$

$$ds^2 = -(R^2 - 1)dT^2 + \frac{dR^2}{R^2 - 1} + R^2 d\Phi^2$$

BTZ Coordinates con't

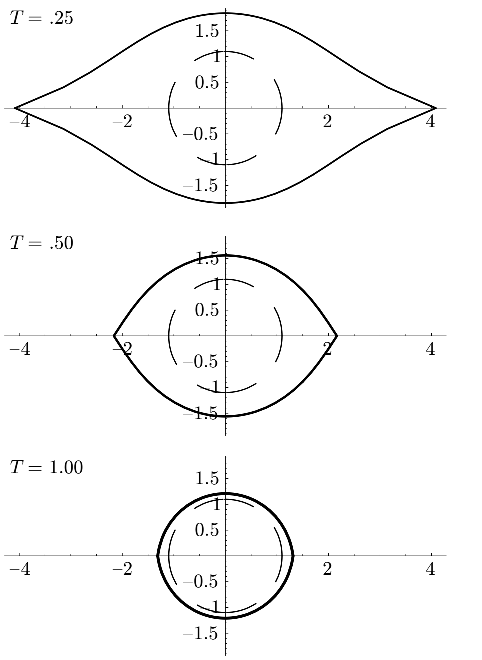
From AdS coordinates:

$$\begin{aligned} x_{-1} &= \sqrt{R^2 - 1}\sinh T \\ x_0 &= R\cosh\Phi \\ x_1 &= \sqrt{R^2 - 1}\cosh T \\ x_2 &= R\sinh\Phi \end{aligned}$$

The geodesic connecting the two particles in CoM frame becomes ($w = \coth\mu/2$):

$$R^2 = \frac{w^2\cosh^2 T}{w^2\cosh^2 T - (\sinh\Phi + w\cosh\Phi)^2}$$

By testing geodesic equation, it is possible to show that this is a space-like geodesic in BTZ coordinates.



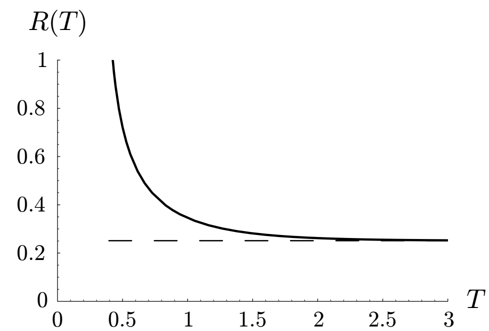
$w = \coth \mu/2 = 2$ Thaler (2002)

Black Hole Formation

Let $R(T)$ be relative distance between particles.

$$R(T) = -\operatorname{arctanh}\left(\frac{1}{\cosh T\sqrt{1+w^2\sinh^2 T}}\right)$$
$$+\operatorname{arctanh}\left(\frac{3+w^2+2w^2\cosh^2 T}{(1+w^2)\cosh T\sqrt{1+w^2\sinh^2 T}}\right)$$

Graph for $w = \coth \mu/2 = 8$:



Thaler (2002)

Relative Hamiltonian

Hamilton's Equation of Motion:

$$\frac{\partial H}{\partial P} = \dot{R}$$

Looking for an expression for H that yields appropriate expression for $\dot{R}$ in the limit $R \to \mu$ ($T \to \infty$).

$$R(T) \approx \mu + 4\cosh 2\mu \tanh\frac{\mu}{2}e^{-2T}$$
$$\dot{R}(T) \approx -8\cosh 2\mu \tanh\frac{\mu}{2}e^{-2T}$$

Key realization: The total energy of the system is the mass of the Black Hole!

$$H = M \propto \mu^2$$

Thus, we can relate R and $\dot{R}$ by:

$$\dot{R} = \frac{\mu^2 - R^2}{\mu} = \frac{H - R^2}{\sqrt{H}}$$

Relative Hamiltonian con't

Try the Hamiltonian:

$$H = R^2\tanh^2\frac{P}{2R}$$

$$\begin{aligned}\frac{\partial H}{\partial P} &= 2R^2\tanh\frac{P}{2R}\operatorname{sech}^2\frac{P}{2R}\frac{1}{2R}\\ &= R\tanh\frac{P}{2R}\left(1-\tanh^2\frac{P}{2R}\right)\\ &= R\left(\frac{-\sqrt{H}}{R}\right)\left(1-\frac{H}{R^2}\right) = \frac{\sqrt{H}}{R^2}(H-R^2)\\ &\approx \frac{H-R^2}{\sqrt{H}} \quad (R^2 \to H = \mu)\end{aligned}$$

This is the expression we were looking for. Last step: calculate the tunneling quantity.

Momentum Behavior Near Horizon

Expanding Hamiltonian for large P:

$$H - D^2 \approx -4D^2\, e^{\frac{P}{2D}}$$

$$P \approx 2D \ln \frac{D^2 - H}{4D^2}$$

Compare to case of test particle in black hole background. There we had linear divergence of momentum.

Here we have a milder *logarithmic* divergence.

Suggests difference in behavior of $\overline{S}$.

Calculating $\overline{S}$

$$\overline{S} = \int^{\mu} P dR$$

For our approximate Hamiltonian:

$$P = -2R \operatorname{arctanh} \frac{\mu}{R}$$

Evaluating the integral ($z = \mu/R$):

$$\int^{\mu} P dR = 2\mu^2 \int^{1} \frac{dz}{z^3} \operatorname{arctanh} z = -\mu^2.$$

The tunneling quantity is *finite*, so for our model of Black Hole Formation, it cannot be a tunneling phenomenon!

In other words:

$$A = e^{\frac{i}{\hbar}\overline{S}} = e^{-\frac{i}{\hbar}\mu^2}$$

$$\text{Probability} = |A|^2 = 1$$

Looking Forward

We now have a simple model of black hole formation.

Easy to visualize (geometry).

Easy to calculate (algebra).

Possible Directions:

1. More General Collisions
2. Exact Hamiltonian
3. Quantum Mechanics

Further Information

References:

Hans-Jürgen Matschull. (1999) *Black hole creation in* $2+1$ *dimensions.* Class. Quantum Grav. **16**, 1069-1095.

Antal Jevicki and Jesse Thaler, *Dynamics of Black Hole Formation in an Exactly Solvable Model*, hep-th/0203172.

Notes:

http://www.jthaler.net/physics/

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