Model of Black Hole Formation in $2 + 1$ Dimensions

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Contents

Chapter 1

An Overview of Black Hole Formation

1.1 The Challenge of Gravity

Einstein's theory of General Relativity stands as one of the most beautiful theories in theoretical physics. Not only is it an extremely accurate description of Gravity, it relies on very few assumptions to develop a robust classical field theory. In $2+1$ dimensions (two space and one time), the theory is exactly solvable, such that given an energy distribution, the curvature of space-time can be calculated without approximation. However, in $3+1$ dimensions (the supposed dimensionality of our space-time), there are very few analytic solutions. In fact, the seemingly simple problem of two body motion is not exactly solvable in $3+1$ dimensional gravity.

One of the features of General Relativity is the appearance of Black Holes. Black holes are a solution to the Einstein equation when given a sufficiently massive lump of matter. If such a mass distribution exists in the universe, we can calculate the curvature of space-time around it. But presumably, a sufficiently massive lump of matter could be created by two or more insufficiently massive lumps joining together. If we could understand this process, then we could understand the process of black hole formation, but because we cannot solve the problem of two body interactions in $3+1$ dimensional gravity, we are forced to make critical assumptions that might skew our notions of black hole formation.

In particular, there has been the suggestion that black hole formation would be exponentially suppressed in a semi-classical model [2]. Models of black hole formation that assume a test particle falling in a black hole background look remarkably similar to Hawking radiation. Because Hawking radiation is exponentially suppressed in semi-classical calculations, the obvious conclusion would be that black hole formation would also be exponentially suppressed.

But a model of black hole formation based on a test particle falling in a black hole background seems arbitrary at best and most likely very wrong. The total energy for black hole formation is the mass of the created black hole. (A simple application of energy conservation and $E = mc^2$

should be sufficient to convince you of this fact.) But test particles are assumed to have negligible kinetic energy, therefore we are forgetting about a potentially huge source of energy by assuming that all of the energy is carried by the black hole background.

As we will see when we make some calculations at the end of the paper, it is the fact that the total of energy of black hole formation is the mass of the created black hole that turns black hole formation from being exponentially suppressed to being neither suppressed nor enhanced. Of course, we will be working in $2+1$ dimensions, and it is unclear whether such calculations can be generalized to $3+1$ dimensions. Still, it appears that at the semi-classical level, black hole formation is less like Hawking radiation and more like playing billiards.

1.2 Implications for Experiment, Opportunities for Theory

One of reasons why we care about exponential suppression is that as particle accelerators reach higher and higher energies, there is a possibility of black hole formation through two particle interactions. Without exponential suppression, a black hole should form when the two particles come within some effective horizon. This assumption is the foundation for phenomenological work in determining the effect of extra dimensions on rates of black hole formations at accelerators [1]. We call these models geometric models, in that they predict black hole formation based on geometric collision cross-sections.

But if there were exponential suppression of black hole formation, then predictions from geometric models would be wildly inaccurate. Even in very high energy collisions, exponential suppression would dominate and very few black hole events would be observed. Therefore, a theoretical understanding of black hole formation is necessary to determine whether or not we need to account for exponential suppression factors. As mentioned above, our exactly solvable model does not exhibit exponential factors, suggesting that predictions from geometric models are reasonable.

Another theoretical reason for studying an exactly solvable model is that we will be able to write down an expression for the Hamiltonian. Given such a Hamiltonian, we can compare with predictions from more complicated theories such as M-Theory. A proposed black hole Hamiltonian from M-Theory will be more reasonable if it takes on the same form as a known exactly solvable Hamiltonian. Also, given a Hamiltonian we can quantize the system, in effect providing a very crude version of quantum gravity.

Thus, our simple model in $2+1$ dimensions is motivated by the search for fundamental properties of black hole formation. If we can understand whether or not black hole formation is suppressed, we will be able to know whether or not to expect black hole events in accelerators. If we can find an expression for a black hole formation Hamiltonian, we can try to recreate such a Hamiltonian in more complicated theories.

1.3 Semi-Classical Transition Amplitudes

In Feynman path integrals, we can calculate the amplitude for a quantum process by summing over all paths the quantity $e^{iS/\hbar}$, where S is the action for a given path. In the semi-classical approximation, we only use the classical path, such that the amplitude is approximately

$$A \approx e^{\frac{i}{\hbar}S_{classical}}. \tag{1.1}$$

The transition probability is given by the magnitude of A squared, so for a finite real-valued classical action, the transition probability is 1. In other words, there is a 100% chance that the classical process will occur. This is not particularly interesting, and one might ask how the semi-classical probability could ever not be 1 considering that $S_{classical}$ is always real in non-tunneling events.

The answer is that $S_{classical}$ could be infinite. In that case, we can perform an $i\epsilon$ deformation using complex analysis and extract an imaginary component of the classical action. For Hawking radiation, the imaginary component gives the temperature of the radiation, which is another way of saying that it calculates the suppression factor for the process of particles leaving the black hole [3]. For black hole formation, if $S_{classical}$ were infinite, then the imaginary component would tell us the exponential suppression (or enhancement) factor for the process.

Calculating $S_{classical}$ is quite simple. We know that the action is the integral of the Lagrangian over time. The Lagrangian is also given by $L = P\dot{R} - H$ where H is the Hamiltonian, so we have

$$S_{classical} = \int (P\dot{R} - H)dt = \int PdR - ET. \tag{1.2}$$

In the last step, we used the fact that the Hamiltonian is the energy of the system and energy is conserved.

We are looking for possible divergences in the action. Clearly as $T \to \infty$, the ET factor diverges, but this would be true of any energy conserving process so it is not very interesting. The other possibility is that P diverges for some value of R. We will see later on that at the horizon of the effective black hole, P does diverge, so we are really only concerned with the quantity

$$\overline{S} = \int^{\mu} PdR \tag{1.3}$$

where μ is the R coordinate of the horizon. The reason that the integral does not have a lower bound is that for any finite value of R greater than μ, P is well-behaved and thus there is no possibility for a divergence.

In order to calculate $\overline{S}$, we need to express P as a function of R near the horizon. In the black hole model that we will use, we will find an expression of R as a function of T, so we will be able to calculate $\dot{R}$ as a function of T. In fact, we will be able to go further and express $\dot{R}$ as a function of R. From Hamilton's equations of motion, we know that

$$\dot{R} = \frac{\partial H}{\partial P}. \tag{1.4}$$

Assuming that we know $\dot{R}$ as a function of R, we should be able to solve for P as a function of R and H. Another ways of saying this, is that we can guess a form of the Hamiltonian and check to see if it satisfies Hamilton's equations of motion. If so, then we will be able to extract P as a function of R and evaluate $\overline{S}$.

1.4 The Origin of Exponential Suppression

Before describing our $2+1$ dimensional black hole model, let us look at the reasons why other models of black hole formation could suggest exponential suppression. A typical black hole in $2+1$ dimensions is the BTZ black hole (which is also the black hole we will create in our model). It is given by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}, \tag{1.5}$$

where $f(r) = r^2 - r_0^2$ and we have ignored the angular part. This leads to the action

$$S = -m \int dt \sqrt{f(r) - \frac{\dot{r}^2}{f(r)}}, \tag{1.6}$$

and the associated Lagrangian

$$L = -m \sqrt{f(r) - \frac{\dot{r}^2}{f(r)}}. \tag{1.7}$$

We would like to turn this expression into a Hamiltonian. Furthermore, to focus on massless particles, we will go to the limit of $m \to 0$. The momentum conjugate to r is

$$p = \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{f(r)\sqrt{f(r) - \frac{\dot{r}^2}{f(r)}}}. \tag{1.8}$$

Solving for $\dot{r}$, we find

$$\dot{r}^2 = \frac{f(r)^3 p^2}{m^2 + f(r)p^2}. \tag{1.9}$$

The Hamiltonian is given by $H = p\dot{r} - L$. As $m \to 0$,

$$L \to 0 \text{ and } \dot{r} \to \pm f(r). \tag{1.10}$$

Note that near the horizon, $f(r) \to 0$, therefore $\dot{r} \to 0$. This is a general feature of particles near black hole horizons and we will see this again when we consider our model of black hole formation. Often, when the velocity goes to zero, the momentum also goes to zero, but this is not the case with black holes. For a massless test particle, we can combine the above results to show that the

Hamiltonian is

$$H = f(r)|p|. \tag{1.11}$$

The absolute value on p absorbs the $\pm$ in the expression for $\dot{r}$ such that incoming particles have negative momenta and outgoing particles have positive momenta. Therefore, for an incoming particle,

$$p = \frac{H}{r_0^2 - r^2} = \frac{H}{(r_0 + r)(r_0 - r)} \approx \frac{H}{2r_0(r_0 - r)}, \tag{1.12}$$

where the last approximation is for $r \approx r_0$. Thus, the momentum diverges linearly as the particle reaches the horizon. When we go to calculate $\overline{S}$, this linear diverge of the momentum will yield an infinity in the integral. Noting that energy is conserved,

$$\overline{S} = \int^{r_0} p dr = \int^{r_0} \frac{H}{f(r)} dr = E \int^{r_0} \frac{dr}{r_0^2 - r^2} = \frac{E}{r_0} \operatorname{arctanh} \frac{r}{r_0}\bigg|^{r_0} = \frac{E}{r_0} \operatorname{arctanh} 1 = \infty. \tag{1.13}$$

Though we won't carry out the $i\epsilon$ deformation, the fact that $\overline{S}$ is infinite when considering a particle in a black hole background suggests why other models of black hole formation might conclude that black hole formation has exponentially factors. The reason why these models predict suppression rather than enhancement has to do with the specifics of the model.

When we carry out the same calculation for our model, we will see that the equations of motion are very similar to a massless particle falling in a black hole background. However, when we go to find the Hamiltonian, we arrive a very different result, which changes the divergence properties of p near the horizon.

1.5 Looking Ahead

The remainder of the paper is divided into four chapters. Chapters 2 and 3 discuss gravity in $2 + 1$ dimensions with zero cosmological constant. Because black holes cannot form without a cosmological constant, these chapters can be skipped without missing the main findings of the paper. Chapter 2 starts with the Einstein equation to understand the space-time surrounding point particles and uses mainly geometric arguments to discuss moving particles and collisions. Chapter 3 uses the more robust method of holonomies to discuss the same problem, and because holonomies are necessary when we introduce a cosmological constant, this chapter is a good introduction to the $SL(2)$ group.

Chapter 4 covers gravity with a negative cosmological constant. Using the method of holonomies developed in chapter 3, we are able to describe head-on particle collisions in a space which is an anti-de Sitter space minus some geodesic wedges. We will be particularly interested in the case of two sufficiently energetic massless particles ("photons") because we will be able to see the formation of a black hole. In chapter 5, we will apply a suitable coordinate transformation to this model of black hole formation in order to find the relative Hamiltonian describing black hole formation. We can use this information to calculate $\overline{S}$. We find that $\overline{S}$ is finite, therefore black hole formation cannot be exponential suppressed semi-classically.

Chapter 2

Point Particles in $2+1$ Dimensions

2.1 Gravity, Simplified

One way to study General Relativity is to look at its properties in lower dimensional spaces. Whereas $3+1$ dimensional gravity is exactly solvable in only the most symmetric cases, $2+1$ dimensional gravity admits quite simple solutions especially in the case of point particles. In this chapter, we will explore the properties of $2+1$ dimensional gravity with zero cosmological constant. Once we understand the geometry of space-time when we introduce stationary or moving particles, we will be able to derive an expression for the relative Hamiltonian of two particles. This calculation anticipates a calculation that we will carry out for the black hole formation process.

Note that black holes are not formed from particle collisions in $2+1$ dimensions with zero cosmological constant. A negative cosmological constant is necessary to allow for black hole formation and we will consider this possibility in subsequent chapters. Most of the results of this chapter follow from simple geometry and Special Relativity. Though a full treatment of this problem would require General Relativity, we will only derive the most basic results. In this way, we can arrive at an understanding of gravity without having to understand the details of space-time curvature.

2.2 One Stationary Particle

In $2+1$ dimensions, the curvature tensor takes on a simple form. In particular, when the Einstein tensor $G^{\nu}_{\mu} \equiv R^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}R$ is identically zero, the curvature tensor $R^{\alpha\beta}_{\lambda\sigma}$ is identically zero. This is not true in $3+1$ dimensions, which is why $3+1$ dimensions can support gravitational waves among other things. But in $2+1$ dimensions, there are so few indices that G^{ν}_{μ} is closely related to $R^{\alpha\beta}_{\lambda\sigma}$.

From the Einstein equation, we know that $G^{\nu}_{\mu} = 8\pi G T^{\nu}_{\mu}$, so when the stress-energy tensor is zero (and assuming the cosmological constant is zero), we have flat space-time. Adding stationary point sources to our space will only introduce topological changes, and away from the sources, space will remain flat [4]. Contrast this with the model of gravity in $3+1$ dimensions where space-time is asymptotically flat far away from point sources but exhibits curvature in the vacuum between the point sources.

It turns out that in the $2+1$ dimensional static case, there is no spatial dependence to the time component of the metric, so we can write the metric as

$$g_{\mu\nu} = \mathrm{diag}(-1, \phi(\mathbf{r}), \phi(\mathbf{r})). \tag{2.1}$$

This is known as the isotropic coordinates, and it is always permissible in two spatial dimensions. Calculating the Christoffel symbols, it is straightforward to show that the curvature scalar is

$$R = -\nabla^2 \ln \phi. \tag{2.2}$$

Also, it will be helpful to have the time-time component of the Einstein tensor:

$$G^0_0 = R^0_0 - \frac{1}{2}\delta^0_0 R = -\frac{1}{2}R = \frac{1}{2}\nabla^2 \ln \phi. \tag{2.3}$$

Let us solve the Einstein equation for the case of one particle of mass m placed at the origin. The only non-vanishing component of the stress-energy tensor is

$$T^{00} = m\delta^2(\mathbf{r}) \tag{2.4}$$

and lowering indices we find

$$T^0_0 = g_{0i}T^{0i} = -m\delta^2(\mathbf{r}). \tag{2.5}$$

Plugging into the Einstein equation and using the fact that $\nabla^2 \ln r = 2\pi\delta^2(\mathbf{r})$:

$$\begin{aligned} G^0_0 &= -8\pi G m \delta^2(\mathbf{r}) \\ \frac{1}{2}\nabla^2 \ln \phi &= -4Gm\nabla^2 \ln r \\ \ln \phi &= -8Gm \ln r \\ \phi &= r^{-8Gm} \end{aligned} \tag{2.6}$$

The other components of the Einstein equation merely tell us that our choice of $g_{00} = -1$ is allowed.

To see what kind of space this is, we can look at the line element

$$ds^2 = -dt^2 + \phi(dr^2 + r^2 d\theta^2) = -dt^2 + r^{-8Gm}(dr^2 + r^2 d\theta^2). \tag{2.7}$$

Let $\rho = \frac{1}{\alpha}r^\alpha$ and $\varphi = \alpha\theta$ where $\alpha = 1 - 4Gm$. The line elements transforms to

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2. \tag{2.8}$$

This looks like flat (Minkowski) space, except that φ runs from 0 to $2\pi\alpha$. In other words, the mass has effectively "cut out" a wedge of angle $8\pi Gm$. We will refer to this angle as the deficit angle. For the remainder of this chapter, we will use coordinates in which $G = \frac{1}{4\pi}$, so the deficit angle becomes $2m$.

2.3 One Moving Particle (Geometric Approach)

We could again solve the Einstein equation using the stress-energy tensor for a moving point particle, but because our space is a Minkowski space minus a wedge, we can use results from Special Relativity. This has the advantage of being more geometrically intuitive than the tensor approach.

Instead of talking about the velocity of a particle, we will use *rapidities*. Given a velocity v, the rapidity ξ is given by

$$\tanh \xi = v. \tag{2.9}$$

Rapidity notation has the advantage that certain quantities from Special Relativity have a simple form. In particular:

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh \xi, \qquad \frac{v}{\sqrt{1-v^2}} = \sinh \xi. \tag{2.10}$$

Consider a particle moving to the right with rapidity ξ. In the rest frame of the particle, align the wedge edges such that they make an angle of m with the horizontal. The slope of these edges are $\pm \tan m$. In a frame moving to the left with rapidity ξ, the slopes will appear to be

$$\tan \beta = \cosh \xi \tan m. \tag{2.11}$$

This follows from recalling that distances in the x-direction are contracted by a factor of $1/\gamma$, but distances in the y-direction are unaffected. Note that for small masses, $\tan m \approx m$, and we recover the result from Special Relativity

$$\beta = m' = \gamma m. \tag{2.12}$$

It is important that the edges were symmetric with respect to line of travel, otherwise distances along the edge would not transform properly. In particular, if we consider points along each wedge that are a distance r from the particle, when we move to the moving frame, both should be a distance r' away. If the wedge edges were not symmetric with respect to the line of travel, the distances would not be preserved.

Another important quantity is the edge rapidity η. It is related to the particle rapidity by simple geometry,

$$\tanh \eta = \sin \beta \tanh \xi. \tag{2.13}$$

This follows from noting that $\tanh \xi$ is the particle velocity, thus the component of the velocity perpendicular to the edge, $\tanh \eta$, will be smaller by a factor of $\sin \beta$. Two other equations that can be derived through geometry are

$$\cos m = \cos\beta \cosh\eta, \tag{2.14}$$

$$\sinh\eta = \sin m \sinh\xi. \tag{2.15}$$

These can be derived from eliminating the appropriate variable from the other two relations. These formulas will become useful when we look at two particle motion.

2.4 One Moving Particle (Algebraic Approach)

Another way to talk about deficit angles is to use the concept of *holonomies*. Whenever we come near a wedge edge, our coordinates have to rotate an angle $2m$ in the rest frame of the particle or an angle 2β in a moving frame. We can imbed this information in a holonomy matrix and use algebraic methods to understand how the holonomies change in different reference frames. The advantage of the holonomy method is that it easily incorporates Lorentz transformations.

In the rest frame of the particle, the holonomy that defines the wedge is simple. The deficit angle is $2m$, so when we cross an edge, we have to rotate that amount about the origin. If our coordinates are (t, x, y) then the matrix representing this transformation is

$$u_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2m & \sin 2m \\ 0 & -\sin 2m & \cos 2m \end{pmatrix}. \tag{2.16}$$

The edges of the wedge at time t are

$$w_{0\pm} : y = \pm x \tan m \tag{2.17}$$

and we see that

$$u_0 \begin{pmatrix} t \\ x \\ x\tan m \end{pmatrix} = \begin{pmatrix} t \\ x \\ -x\tan m \end{pmatrix}. \tag{2.18}$$

The Lorentz boast for a frame moving to the right at a rapidity ξ is

$$\Lambda = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 \\ \sinh\xi & \cosh\xi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.19}$$

So if we are in a frame in which we see a particle moving to the right at a rapidity ξ, the holonomy we observe will be

$$u = \Lambda u_0 \Lambda^{-1}. \tag{2.20}$$

This represents moving into the rest frame of the particle, taking the holonomy, and then moving back into the original frame. To find the holonomies when the particle is not at the origin at time $t = 0$, we can perform the appropriate translations. For our purposes, it will not be necessary to consider this general case.

Let us calculate u to show that the deficit angle is indeed what we expect. The edges of the wedge at time $t = 0$ should be

$$w_\pm : y = \pm(x - t\tanh\xi)\tan\beta. \tag{2.21}$$

Recall that $\tanh\xi$ is the velocity of the particle, thus the particle trajectory is $y = 0$, $x = t\tanh\xi$.

Calculating u we find

$$u = \begin{pmatrix} \cosh^2\xi - \sinh^2\xi\cos 2m & \cosh\xi\sinh\xi(\cos 2m - 1) & \sinh\xi\sin 2m \\ \cosh\xi\sinh\xi(1-\cos 2m) & \cosh^2\xi\cos 2m - \sinh^2\xi & \cosh\xi\sin 2m \\ \sinh\xi\sin 2m & -\cosh\xi\sin 2m & \cos 2m \end{pmatrix}. \tag{2.22}$$

Using the fact that $\tan\beta = \cosh\xi\tan m$, we can calculate that

$$u\begin{pmatrix} t \\ x \\ (x - \tanh\xi)\tan\beta \end{pmatrix} = \begin{pmatrix} t \\ x \\ -(x - \tanh\xi)\tan\beta \end{pmatrix}. \tag{2.23}$$

Note that we could have solved for y if we had assumed that u sent (t, x, y) to $(t, x, -y)$. In some sense, this is an amazing result. In the geometric approach, we were able to find the deficit angle using some knowledge of Special Relativity. In the algebraic approach, we not only found the deficit angle, but we also found the particle trajectory and the equations for the wedge edges for all t given only the holonomy. In fact, it is conceivable that a simple formula could give us the holonomy, and then all of the physics of the problem could be derived from this holonomy. We will explore this possibility in the next chapter, and use it in the following chapter to discuss a space-time that is not so geometrically intuitive.

2.5 Two Particles

Because masses only affect space by cutting out wedges, it is easy to incorporated multiple particles in $2 + 1$ dimensions: each particle m_i cuts out a wedge of deficit angle $2\beta_i$. Assuming the total mass is not large enough to close space (in which case you have to worry about image masses), the qualitative description of multiple masses in $2 + 1$ dimensional gravity is Minkowski space with multiple excisions.

Just like for one particle, we can use algebraic or geometric approaches to describe two particles. The advantage of the algebraic (i.e. holonomy) approach is that generalizes more easily to multiple particles. The disadvantage is that it requires an understanding of the $SL(2)$ group, which we will develop in the next paper. In the meantime, for sufficiently symmetric cases, a geometric approach is all that is necessary.

We will explore the head on collision of two particles of mass m. Let the first mass be stationary at the origin and let the second mass be traveling in the $+x$ direction with rapidity ξ. There seems to be an asymmetry in this frame of reference for though the masses are the same, one particle is moving and the other is not. We would like to move to the center-of-mass frame in order to isolate the relative motion of the particles, but is unclear what "center-of-mass" means in the context of gravity.

Consider the following construction. Make a cut along the x-axis and rotate the two halves an angle m in order to close the deficit angle on the right side of the first particle. Note that the second particle now has two locations in the plane, but because the wedge edges are identified, they actually represent the same point in space. Now extend the wedge edge of the second particle until it meets the x-axis. We will call this the center-of-mass of the system. Except for an extra excised triangle, this looks like an effective mass at the center-of-mass cutting out an effective deficit angle.

If we put the second particle in motion, we notice that the center-of-mass changes its location. Thus, to move to the center-of-mass frame in which the center-of-mass is stationary, we want to perform a Lorentz transformation such that the second particle motion is in the same direction as the effective deficit angle.

Before we performed the rotation, the second particle motion was given by

$$(t, x, y) = (\lambda \cosh \xi, \lambda \sinh \xi, 0), \tag{2.24}$$

where λ is proper time which runs from $-\infty$ to 0. After the rotation by m to close the first partcle's right side deficit angle, the second partilce motion is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos m & \sin m \\ 0 & -\sin m & \cos m \end{pmatrix} \begin{pmatrix} \lambda \cosh \xi \\ \lambda \sinh \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \cosh \xi \\ \lambda \sinh \xi \cos m \\ -\lambda \sinh \xi \sin m \end{pmatrix}. \tag{2.25}$$

Now consider a boost by a rapidity ζ_1 in the $-x$ direction. (The rapidity is called ζ_1 because it is the rapidity of the first particle after the Lorentz boost.) The boosted second particle motion is:

$$\begin{pmatrix} \cosh \zeta_1 & -\sinh \zeta_1 & 0 \\ -\sinh \zeta_1 & \cosh \zeta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \cosh \xi \\ \lambda \sinh \xi \cos m \\ -\lambda \sinh \xi \sin m \end{pmatrix} = \begin{pmatrix} \lambda \cosh \xi \cosh \zeta_1 - \lambda \sinh \xi \cos m \sinh \zeta_1 \\ -\lambda \cosh \xi \sinh \zeta_1 - \lambda \sinh \xi \cos m \cosh \zeta_1 \\ -\lambda \sinh \xi \sin m \end{pmatrix} \tag{2.26}$$

We want this vector to equal motion along the wedge edge at a rapidty ζ_2:

$$(t, x, y) = (\lambda \cosh \zeta_2, \lambda \sinh \zeta_2 \cos M, -\lambda \sinh \zeta_2 \sin M), \tag{2.27}$$

where $M = \beta_1 + \beta_2$ is an effective mass. Setting these particle motions equal to each other we find

$$\sinh \zeta_2 = \frac{\sinh \xi \sin m}{\sin M}, \qquad \sinh \zeta_1 = \frac{\sinh \xi \sin m}{\sin M}. \tag{2.28}$$

We would like to know what this M is in terms of m and ξ, the original parameters defining the problem. Note that $\zeta_1 = \zeta_2 \equiv \zeta$, and because of the one particle result that $\tan\beta_i = \cosh\zeta_i \tan m$, we see that $\beta_1 = \beta_2 \equiv \beta$. Using the relation $\sinh\zeta = \sinh\xi \sin m / \sin M$ and $M = 2\beta$ it is possible to show that

$$\cos M = \cos^2 m - \sin^2 m \cosh\xi. \tag{2.29}$$

In the next chapter, we will use the holonomy approach to derive a more general form of this equation. One way to check this formula is to make sure that it makes sense in the case of two moving particles. Consider particle one with rapdity ξ_1 to the right and particle two with rapidity ξ_2 to the left. From Special Relativity, the relative rapidity ξ is given by

$$\tanh\xi = \frac{\tanh\xi_1 + \tanh\xi_2}{1 + \tanh\xi_1 \tanh\xi_2}. \tag{2.30}$$

We can find $\cosh\xi$ by a trigonometric identity.

$$\begin{aligned} \cosh\xi &= \sqrt{\frac{1}{1-\tanh^2\xi}} = \frac{1 + \tanh\xi_1 \tanh\xi_2}{(1-\tanh^2\xi_1)(1-\tanh^2\xi_2)} \\ &= \cosh\xi_1 \cosh\xi_2 + \sinh\xi_1 \sinh\xi_2 = \cosh(\xi_1 + \xi_2) \end{aligned} \tag{2.31}$$

So in the formula for $\cos M$, rapidities add as expected.

2.6 Two Particle Relative Hamiltonian

The key realization for finding the two particle relative Hamiltonian is that the effective mass of the two particles *is* the total energy of the system. We already have an expression for $\cos M$; all we need to do is find a coordinate/momentum pair to describe the relative motion of the two particles. Clearly the coordinate should be the distance between the particles, but what about the momentum?

The momentum should be a type of velocity. At first, we might think of the parameter ξ, but this was a rapidity in a coordinate system with one particle at rest. Then, we might think of ζ, the rapidity in the center-of-mass frame, but when we calculate $\frac{\partial H}{\partial \zeta}$ we do not find the expected expression for $\dot{R}$.

It turns out that the variable conjugate to the relative distance is the edge rapidity η. Recall from the geometric approach that the edge radidity is given by $\sinh\eta = \sin m \sinh\zeta$. (We are using ζ because this is the rapidity of the particles in the center-of-mass frame.) Plugging in the expression for $\sinh\zeta$,

$$\sinh\eta = \frac{\sin^2 m \sinh\xi}{\sin M}. \tag{2.32}$$

Now we will eliminate ξ from the expression for $\cos M$.

$$(\cos^2 m - \cos M)^2 = \sin^4 m(1 + \sinh^2\xi) = (\sin^4 m + \sinh^2\eta \sin^2 M) \tag{2.33}$$

Rearranging, and letting $H = M$:

$$\cosh \eta = \cos m \sqrt{\frac{2}{1+\cos H}} = \frac{\cos m}{\cos \frac{H}{2}}$$

$$\cos \frac{H}{2} = \frac{\cos m}{\cosh \eta} \tag{2.34}$$

Calculating $\dot{R} = \frac{\partial H}{\partial \eta}$:

$$-\frac{1}{2} \sin \frac{H}{2} \frac{\partial H}{\partial \eta} = \frac{\cos m}{\cosh^2 \eta} \sinh \eta$$

$$\frac{\partial H}{\partial \eta} = \frac{2 \tanh \zeta}{\tan \frac{H}{2}} \tag{2.35}$$

Finally, we need to calculate $\dot{R}$ directly to check for agreement.

$$\begin{aligned} \dot{R} &= \vec{v_1} - \vec{v_2} = \sqrt{(\tanh \zeta + \tanh \zeta \cos H)^2 + \tanh^2 \eta \sin^2 H} \\ &= \tanh \zeta \sqrt{2 + 2\cos H} = 2 \tanh \zeta \cos \frac{H}{2} \end{aligned} \tag{2.36}$$

Recall that $H/2 = \beta$ and the relation $\tanh \eta = \sin \beta \tanh \zeta$.

$$\dot{R} = \frac{2 \tanh \eta}{\tan \frac{H}{2}} \tag{2.37}$$

This is the relation we were looking for. If we wanted to consider the case of unequal masses, we could repeat the calculation and we would find that the edge rapidity was once again conjugate to the relative distance. (The algebra for this case is difficult but doable). To consider a system with angular momentum (i.e. two particles who do not collide head-on), we would have to introduce an angular variable and momentum pair, but it is unclear how to go about this without further analysis.

2.7 The Promise of Holonomies

As one might guess, when we consider less symmetric systems, it will become more difficult to use geometric approaches. The algebraic approach of holonomies will be very useful when looking at more general interactions of two or more particles. In addition, holonomies will help us to understand gravity in spaces with a cosmological constant for which Special Relativity alone is inadequate. We will explore this in the next two chapters.

What you should take away from this chapter is that to understand particles in 2 + 1 dimensions, we need not consider the full formulation of General Relativity. Instead, we can concentrate on geometric or algebraic results, making it far easier to visualize gravity.

Chapter 3

Holonomy Approach to Gravity

3.1 From Geometry to Algebra

In the previous chapter, we looked at point particles in $2+1$ dimensional gravity and used a geometric approach to find the relative Hamiltonian of two point particle in the highly symmetric case of equal masses and zero angular momentum. In this chapter, we would like to use the algebraic approach of holonomies to tackle the problem of different masses and even massless particles. In order to do so, we will have to introduce the algebra of $SL(2)$. We have another reason to discuss $SL(2)$, however. In the next chapter, it will allow us to tackle the problem of point particles in a space with non-zero cosmological constant.

Note that this chapter will not deal with the more general case of non-zero angular momentum. The reason is that in order to consider particles whose world lines do not intersect, we need to know how to describe translations in $2+1$ dimensions. Whereas rotations and Lorentz transformation are linear operations in flat space, translations are not. For the more general two particle interactions see [5]. Because we will only be interested in the formation of a non-rotating black hole, the case of zero angular momentum is sufficient.

3.2 $SL(2)$

$SL(2)$ is the group of 2×2 real matrices with determinant 1. The group $SL(2)$ is isomorphic to $O(2,1)$, the group of rotations in Minkowski space. Thus, the elements of $SL(2)$ are in one-to-one correspondance with the Lorentz transformations. The main reason for using $SL(2)$ is that it is easier to manipulate 2×2 than 3×3 matrices.

We will also be interested in the Lie Algebra that generates $SL(2)$. We will call this algebra $sl(2)$.

From group theory, we know that every element of $SL(2)$ can be written as the exponential of an element of $sl(2)$. $sl(2)$ consists of the real traceless 2×2 matrices spanned by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

Thus, every element $u \in SL(2)$ can be written as

$$u = e^{q^a \gamma_a}, \tag{3.2}$$

where q^a are real coefficients. The generating matrices satisfy the multiplication rule

$$\gamma_a \gamma_b = g_{ab} \mathbf{1} - \epsilon_{ab}{}^c \gamma_c, \tag{3.3}$$

where $g_{ab} = \text{diag}(-1, 1, 1)$ is the metric for $2+1$ dimensional Minkowski space, and ϵ^{abc} is the Levi-Civita symbol.

To see why $SL(2)$ is isomorphic to the rotations in Minkowski space, consider its action on Minkowski coordinates. We can imbed the coordinates (t, x, y) into a matrix:

$$z = t\gamma_0 + x\gamma_1 + y\gamma_2 = x^a \gamma_a. \tag{3.4}$$

(This matrix is *not* an element of $SL(2)$ but is an element of $sl(2)$.) When one matrix acts on another, we multiply on the right by the matrix and on the right by the inverse of the matrix. Let us see how $u = e^{-\frac{\theta}{2}\gamma_0}$ acts on z.

$$u^{-1} z\, u = t\gamma_0 + (x \cos\theta - y \sin\theta)\gamma_1 + (y \cos\theta + x \sin\theta)\gamma_2 \tag{3.5}$$

This is precisely a rotation counter-clockwise by an angle θ in the xy plane.

To accomplish a boost, we can try the matrix $u = e^{-\frac{\zeta}{2}\gamma_2}$:

$$u^{-1} z\, u = (t \cosh\zeta + x \sinh\zeta)\gamma_0 + (x \cosh\zeta + t \sinh\zeta)\gamma_1 + y\gamma_2 \tag{3.6}$$

This is a boost in the positive x direction. For the record, a boost in positive y direction is accomplished using $u = e^{\frac{\zeta}{2}\gamma_1}$. (Note the absense of the minus sign.)

Most of the time it is more convient to write elements of $SL(2)$ without exponentials. By expanding the exponential it follows that:

$$u = e^{q^a \gamma_a} = \cos Q + \sin Q\, q^a \gamma_a, \qquad Q = \sqrt{-q^a q_a}. \tag{3.7}$$

When we begin to consider the holonomies of particles in $2+1$ dimension, the mass of the particle will take the place of Q. Thus, if we can find the effective holonomy for a collection of particles, we should be able to find the effective mass of the system. Moreover, q^a will be related to the momentum of the system, so with an effective holonomy we should also be able to find the effective momentum of the system.

3.3 Single Particle Holonomy

What is the holonomy for a single particle look like in $SU(2)$ notation? It has to be a space-like holonomy that rotates an angle $2m$ clockwise, so a good guess would be

$$u_0 = e^{m\gamma_0} = \cos m\,\mathbf{1} + \sin m\,\gamma_0. \tag{3.8}$$

We can check to see if this is correct by applying it to the known wedge edge.

$$z = \lambda(\gamma_0 + \gamma_1 + \tan m\,\gamma_2) \tag{3.9}$$

$$u_0^{-1} z\, u_0 = \lambda(\gamma_0 + \gamma_1 - \tan m\,\gamma_2) \tag{3.10}$$

Clearly, this holonomy only works for a particle which passes the origin at $t = 0$, but through translations, we can move it to any desired point. Note how the $SL(2)$ language allows us to easily write down the holonomy given the mass of the particle.

We can Lorentz boost the stationary particle holonomy to find the holonomy for a moving particle.

$$v = e^{-\frac{\xi}{2}\gamma_2} \tag{3.11}$$

$$u = v^{-1} u_0\, v = e^{m\cosh\xi\gamma_0 + m\sinh\xi\gamma_1} = \cos m\,\mathbf{1} + \sin m\cosh\xi\,\gamma_0 + \sin m\cosh\xi\,\gamma_2 \tag{3.12}$$

Again, we could check to see whether this holonomy maps the wedge edges as expected, but it is more interesting to try to "eliminate" ξ from the expression for the holonomy. We know that for a particle of mass m moving in the x direction with rapidity ξ, the energy-momentum three-vector is

$$(p^0, p^1, p^2) = (m\cosh\xi, m\sinh\xi, 0). \tag{3.13}$$

Plugging this into the expression for the holonomy, we find

$$u = e^{p^0\gamma_0 + p^1\gamma_1}. \tag{3.14}$$

By extension, it seems reasonable that given any energy-momentum three-vector (p^0, p^1, p^2), the holonomy for that particle should be

$$u = e^{p^0\gamma_0 + p^1\gamma_1 + p^2\gamma_2} = e^{p^a\gamma_a} = \cos m\,\mathbf{1} + \frac{\sin m}{m}\,p^a\gamma_a, \tag{3.15}$$

where $m^2 = p_0^2 - p_1^2 - p_2^2$ is the mass of the particle. This form of the holonomy shows explicitly the dependence of the deficit angle on the particle momentum. Whereas in the geometric approach, we had to be careful to align the wedge edges properly before doing Lorentz boosts, in this algebraic approach, we merely have to know the momentum of a particle to find the holonomy.

By writing the holonomy in terms of m, we are able to consider massless particles quite easily. As $m \to 0$, $\cos m \to 1$ and $\sin m/m \to 1$, so the holonomy for a massless particle will be

$$u = \mathbf{1} + p^a\gamma_a, \tag{3.16}$$

where we have the constraint $p_0^2 = p_1^2 + p_2^2$. Using the energy-momentum three-vector more directly, consider a "photon" moving in the x direction with energy $\tan\epsilon$:

$$(p^0, p^1, p^2) = (\tan\epsilon, \tan\epsilon, 0). \tag{3.17}$$

Thus, the holonomy will be

$$u = e^{\tan\epsilon(\gamma_0+\gamma_1)} = \mathbf{1} + \tan\epsilon(\gamma_0 + \gamma_1). \tag{3.18}$$

What is most fascinating about the holonomy approach and what makes it so powerful is that it describes the excised region of space-time *and* the particle trajectories for all times t. We got a sense of this when we first looked at the algebraic approach in the last chapter, but with the $SL(2)$ language, the computations become even simpler.

Consider a photon with energy $\tan\epsilon$ moving the $+x$ direction which passes the origin of space-time ($x = 0$, $y = 0$ at $t = 0$). As we showed above, the holonomy will be

$$u = \mathbf{1} + \tan\epsilon(\gamma_0 + \gamma_1). \tag{3.19}$$

To find the wedge edges, we are looking for curves $w_\pm$ such that

$$u^{-1} w_- u = w_+. \tag{3.20}$$

By the symmetry of the problem about the x axis, we can assume that

$$w_+ = t\gamma_0 + x\gamma_1 + y\gamma_2, \qquad w_- = t\gamma_0 + x\gamma_1 - y\gamma_2. \tag{3.21}$$

Solving these equations (a trivial exercise in matrix multiplication) we find

$$y = (t - x)\tan\epsilon, \tag{3.22}$$

which defines a wedge of angle 2ϵ. The intersection of the wedges will define the particle motion:

$$y = 0, \quad x = t, \tag{3.23}$$

which is a particle moving to the right with velocity 1, just what we expected. We can continue this method for arbitrary masses and velocities, and we recover the major results from the geometric approach with far fewer calculations.

3.4 Effective Holonomy for Two Particles

The other main strength of the holonomy approach is that is gives a simple way to talk about the motion of multiple particles. In particular, the holonomy approach gives a natural definition of the center-of-mass of a system. Unfortunately, the holonomy approach does *not* tell us very much about the relative motion of particles. For our calculations on black hole formation, we are primarily interested in the relative Hamiltonian of two particles. Because the effective holonomy for two particles effectively eliminates all information about the relative motion of the particles, it would seem that holonomies would hinder rather than aid the search for a relative Hamiltonian.

On the other hand, in the last chapter we saw that one of the difficulties of finding the relative Hamiltonian was identifying the center-of-mass of the system. In $2+1$ dimensions with zero cosmological constant we could extend the wedge edges to find what seemed to be a reasonable center-of-mass. When we add a cosmological constant, we will not have this luxury, so we want a more rigorous center-of-mass description.

Given two rotations, we know that the effective rotation will be the product of the rotation. Because holonomies are just rotations in Minkowski space, given two holonomies, is it reasonable to say that the effective holonomy will be the product of the holonomies. Note that the effective holonomy will not be the true holonomy because when the particles are separated by some distance, the specifics of space-time near the particles depends on knowing which holonomy applies in which regions of space-time. But far away from the particles the holonomy will be approximately $u_{\mathit{eff}} \approx u_1 u_2$.

Because every holonomy corresponds to some particle motion, we expect that the effective holonomy should correspond to an effective mass traveling at some effective velocity. More specifically, the particle motion described by the effective holonomy should be motion of the center-of-mass of the system.

Let the particles have mass m_1 and m_2 with energy-momentum three-vectors (p_1^0, p_1^1, p_1^2) and (p_2^0, p_2^1, p_2^2), respectively. By explicit muliplication, we can write the effective holonomy as

$$u_{\mathit{eff}} = \cos M\,\mathbf{1} + P^a \gamma_a, \tag{3.24}$$

$$\cos M = \cos m_1 \cos m_2 + \frac{\sin m_1 \sin m_2}{m_1 m_2} g_{ab} p_1^a p_2^b, \tag{3.25}$$

$$P^a = \frac{\sin m_1 \cos m_2}{m_1} p_1^a + \frac{\cos m_1 \sin m_2}{m_2} p_2^a - \frac{\sin m_1 \sin m_2}{m_1 m_2} \epsilon_{bc}{}^a p_1^b p_2^c. \tag{3.26}$$

Note that P^a is not truly an effective momenta because P^a includes a factor of $\sin M/M$. The algebra to decouple the effective momenta from this mass factor is not worth the effort, especially because if we are looking for a center-of-mass, we want P^1 and P^2 to be zero, regardless of the value of $\sin M/M$.

More interesting is the fact that the expression for P^a changes under an exchange of the particles. This is because $\epsilon_{bc}{}^a$ is anti-symmetric in the lower indices. There is a non-trivial ordering issue with whether to let $u_{\mathit{eff}} \equiv u_1 u_2$ or $u_{\mathit{eff}} \equiv u_2 u_1$. The different orderings must correspond to different coordinate maps of the same space, because they have to represent the same physical reality. Luckily, when we go to the center-of-mass frame, $u_1 u_2 = u_2 u_1$ so no problem arises with the definition of the center-of-mass.

We can check that the formula for $\cos M$ reduces to our previous result for $m_1 = m_2 = m$, $\vec{p}_1 = (m, 0, 0)$, $\vec{p}_2 = (m\cosh\xi, m\sinh\xi, 0)$. In fact, the object $g_{ab} p_1^a p_2^b$ is Lorentz invariant, so the formula for $\cos M$ holds in all reference frames. This means that the holonomy approach gives a simple way to find the energy of two particles in $2+1$ dimensions.

To find the center-of-mass of the system, we want to move to a reference frame for which $P^0 = \sin M$, $P^1 = P^2 = 0$. Let v be a Lorentz transformation. The new holonomies will be:

$$u_1' = v^{-1} u_1\, v, \qquad u_2' = v^{-1} u_2\, v,$$
$$u_{\mathit{eff}}' = u_1' u_2' = v^{-1} u_{\mathit{eff}}\, v. \tag{3.27}$$

Thus, to find the Lorentz transformation that results in the center-of-mass frame, we trying to solve the matrix equation:

$$v^{-1} P^a \gamma_a\, v = \sin M \gamma_0. \tag{3.28}$$

In other words, we want to transform the effective holonomy to eliminate P^1 and P^2.

In general, this is not an easy problem to solve, but in a sufficiently symmetric case, it is straightforward. Consider the slight generalization of the two particle problem we looked at in the last chapter by letting the masses be different:

$$\vec{p}_1 = (m_1, 0, 0), \qquad \vec{p}_2 = (m_2 \cosh \xi, m_2 \sinh \xi, 0). \tag{3.29}$$

The effective “momenta” are:

$$\begin{aligned} P^0 &= \sin m_1 \cos m_2 + \cos m_1 \sin m_2 \cosh \xi, \\ P^1 &= \cos m_1 \sin m_2 \sinh \xi, \\ P^2 &= \sin m_1 \sin m_2 \sinh \xi. \end{aligned} \tag{3.30}$$

We expect the Lorentz transformation that will bring us to the center of mass frame will be a combination of a rotation and a boost in the x-direction. Consider

$$v = e^{-\frac{\theta}{2}\gamma_0} e^{-\frac{\zeta_1}{2}\gamma_2}.$$

(We call the boost rapidity ζ_1 because it will be the rapidity of the first particle after the transformation.) Solving for θ and ζ_1, we find:

$$\theta = -m_1, \qquad \sinh \zeta_1 = -\frac{\sin m_2 \sinh \xi}{\sin M} \tag{3.31}$$

If we define ζ_2 by

$$\sinh \zeta_2 = -\frac{\sin m_1 \sinh \xi}{\sin M}, \tag{3.32}$$

then by considering holonomies of the individual particles, we find their energy-momentum three-vectors in the center-of-mass frame to be:

$$\begin{aligned} \vec{p}_1{}' &= (m_1 \cosh \zeta_1, m_1 \sinh \zeta_1, 0), \\ \vec{p}_2{}' &= (m_2 \cosh \zeta_2, m_2 \sinh \zeta_2 \cos M, m_2 \sinh \zeta_2 \sin M). \end{aligned} \tag{3.33}$$

Note that this is a particle traveling toward the origin from the x direction, and a particle traveling traveling toward the origin from an direction that makes some angle with the x axis. This construction will appear again in our model of black hole formation.

3.5 The Problem of Relative Motion

Though we now have a general formula for the energy of two particles in $2+1$ dimensions and though we also have a prescription for finding the center-of-mass frame for a two particle system, we do not know how to describe the relative motion of the particles. It turns out that the edge rapidity will again be the momentum conjugate to the relative distance coordinate, but it is difficult to show this when the masses of the particle are different.

In the next chapter, we will add a cosmological constant to our $2+1$ dimensional space. Holonomies will play an important role in this new space, but the idea of an edge rapidity makes little sense, so we will have to find a different way to identify the conjugate momentum. The description of relative motion is essential for understanding the process of black hole formation, and we will spend considerable time in the last chapter trying to find a way to describe the relative Hamiltonian. For now, though, let us try to understand the role of holonomies in a space with negative cosmological constant.

Chapter 4

Gravity with a Cosmological Constant

4.1 AdS Space

Anti-de Sitter or AdS space is a vacuum solutions to the Einstein equation with negative cosmological constant. In $2+1$ dimension, we often call the space AdS_3. Though there are many ways to parametrize AdS space, the most useful is the following:

$$x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 = \ell^2, \tag{4.1}$$

where ℓ is related to the cosmological constant. For this chapter, we will work in units where $\ell = 1$.

In the last chapter, we saw that the rotations in Minkowski space could be represented by elements of the group $SL(2)$. It turns out that rotations in AdS space can also be represented by elements of $SL(2)$. More interesting, though, is the fact that the *coordinates* in AdS are also elements of the group $SL(2)$. To see this, we can write a coordinate of AdS space as

$$z = x_{-1}\mathbf{1} + x_0\gamma_0 + x_1\gamma_1 + x_2\gamma_2. \tag{4.2}$$

The condition that the determinant of z equals one (i.e. that $z \in SL(2)$) reduces to

$$x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 = 1, \tag{4.3}$$

which is precisely the constraint that defines AdS space.

There are many other parametrizations of AdS space. Starting with x_i, we can define "cylindrical" coordinates χ, ϕ, and t.

$$\begin{aligned} x_{-1} &= \cosh\chi\cos t, \\ x_0 &= \cosh\chi\sin t, \\ x_1 &= \sinh\chi\cos\phi, \end{aligned}$$

$$x_2 = \sinh\chi\sin\phi. \tag{4.4}$$

The line element $ds^2 = -x_{-1}^2 - x_0^2 + x_1^2 + x_2^2$ becomes

$$ds^2 = -\cosh^2\chi\, dt^2 + d\chi^2 + \sinh^2\chi\, d\phi^2. \tag{4.5}$$

Thus, t is a time coordinate, χ is a radial coordinate, and ϕ is an angular coordinate. Often in the literature, χ is replaced by $r = \tanh\chi/2$, and with this replacement, AdS space is mapped to time-evolving disk of radius $r = 1$, often called the Poincaré disc. For this paper, we will stick with χ because it makes some of the algebra easier.

In the cylindrical coordinates, the group element of $SL(2)$ that defines the coordinate space is

$$\begin{aligned} z &= \cosh\chi(\cos t\mathbf{1} + \sin t\,\gamma_0) + \sinh\chi(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) \\ &= e^{\frac{1}{2}(t+\phi)\gamma_0} e^{\chi\gamma_1} e^{\frac{1}{2}(t-\phi)\gamma_0}. \end{aligned} \tag{4.6}$$

When written in the second form, we see that the coordinates of AdS look like Euler angle rotations. Because both the transformation of AdS space and the coordinates of AdS space are elements of $SL(2)$, the algebra defining holonomies will be quite manageable.

4.2 Gravity and Geodesics

In 2 + 1 dimensions without a cosmological constant, point particles introduced wedge cut-outs from Minkowski space. This is partly due to the fact that in 2 + 1 dimensions, point particles can only introduce topological changes. Recall that Minkowski space was a vacuum solution to the Einstein equation with zero cosmological constant. Because AdS space is a vacuum solution to the Einstein equation with negative cosmological constant, it seems reasonable to expect point particles to introduce similar "wedge" cut-outs from AdS space.

Though we will not prove it here, by applying the Einstein equation with negative cosmological constant to a system with a point mass, it is indeed possible to show this fact. As one might guess, the cut-out regions in AdS space are also determined by holonomies. To find the wedge edges in Minkowski space, we looked for straight lines that were mapped to each other by the holonomy. The generalization of straight lines in curved spaces are geodesics, therefore to determine the wedge edges in AdS space, we will look for geodesics in AdS that are mapped to each other by the holonomy defining the particle.

There are two sets of geodesics in AdS space that will be useful for us. One set of geodesics are the time independent ones defined by

$$\tanh\chi\cos(\phi - \alpha) = \cos\beta, \tag{4.7}$$

where α is called the center of the geodesic and β is the radius of the geodesic. (This terminology comes from interpreting the geodesic as a circle on a Poincaré disc.) These geodesics will represent the wedge edges. The second set of geodesics are the angular independent ones defined by

$$\tanh \chi = \tanh \xi \sin t, \tag{4.8}$$

where ξ is a rapidity variable. These geodesics will represent particle positions, assuming that the particle passes through $\chi = 0$, $t = 0$. The complete derivation for these geodesic equations is given in [6].

4.3 Holonomies, Revisited

The holonomy for a particle in AdS is exactly the same as for a particle in Minkowski space. This is by no means obvious. Because the coordinates in AdS space are now elements of $SL(2)$, we might expect the transformations in AdS space to be elements of a different group. The fact that the coordinates and transformations belong to the same group explains why AdS is such a rich structure and why physicists like to study it. In contrast, de Sitter space, or dS space, is the vacuum with a positive cosmological constant, and the coordinates and transformations do not belong to the same group.

One way to see that the holonomies in flat space should be the same in AdS space is to consider looking at a small enough region of space-time. It will look nearly flat, therefore the holonomy from Minkowski space should apply in this small region. The fact that the holonomy holds globally is a result of solving the Einstein equations explicitly, and except for the argument that a holonomy of the form $e^{p^a \gamma_a}$ is the simplest (and therefore correct) solution, one should refer to the exact derivation in [7].

As we saw in Minkowski space, for a particle at rest at the origin, the holonomy is

$$u_0 = e^{m\gamma_0}. \tag{4.9}$$

For particles not at the origin, we have to do a translation, but translations in AdS space turn out to be elements of $SL(2)$. For our purposes, we can ignore this translation symmetry subgroup, but it turns out to have an effect on the energy of the system. In particular, whereas in Minkowski space we could read off the total energy of the system from the effective holonomy, this will not be true in AdS space.

The reason it is not true may seem mysterious seeing that the result from Minkowski space was achieved through suitable hand-waving with respect to the meaning of the effective holonomy. This hand-waving worked in Minkowski space because there was a separation of space and time in that space translations do not change time coordinates. In AdS space, proper time runs at different speeds depending on the χ coordinate, so conformal χ translations necessarily affect time and therefore the variable conjugate to time, energy.

Thus, the holonomy approach will be able to tell us about the geometry of AdS space, but it will not be able to tell us about the energy of, say, two particles. This will not be of too much concern because the holonomy approach in Minkowski space only told us about total energy, not relative energy, and we know that in order to understand black hole formation we need to find a relative Hamiltonian. Luckily, knowing the geometry of AdS space will allow us to tease out a relative Hamiltonian.

4.4 Photon Collisions

Though it is possible to derive general results for massive particles in AdS space, our model of black hole formation is based on the head-on collision of two massless particles, so we will focus on the holonomies of "photons". The reason for the quotation marks is that electromagnetic radiation cannot be sustained in $2+1$ dimensions so it is not possible for real photons to exist in our space. For our purposes, a photon merely represents a massless particle.

In AdS space, the x direction will be defined by $\phi = 0$, and the y direction by $\phi = \pi/2$. (This is what we expect in cylindrical coordinates.) As we saw in the last chapter, the holonomy defining a photon traveling in the x direction is

$$u = \mathbf{1} + \tan\epsilon(\gamma_0 + \gamma_1). \tag{4.10}$$

This system has a symmetry about the x axis so we will assume symmetry of the wedge edges under the exchange of ϕ with $-\phi$. To find the wedge edges for the holonomy, we can look at a general position in cylindrical coordiantes:

$$w_\pm = \cosh\chi(\cos t\mathbf{1} + \sin t\,\gamma_0) + \sinh\chi(\cos(\pm\phi)\,\gamma_1 + \sin(\pm\phi)\,\gamma_2). \tag{4.11}$$

We want our holonomy to send w_- to w_+. Solving the matrix equation $w_+ = u^{-1}w_-u$, we find

$$\tanh\chi\sin(\epsilon+\phi) = \sin t\sin\epsilon. \tag{4.12}$$

We can compare this equation to the time-independent geodesics discussed earlier in this chapter. They had the form:

$$\tanh\chi\cos(\phi-\alpha) = \cos\beta \tag{4.13}$$

where α was the "center" of the geodesic and β was the "radius" of the geodesic.

$$\alpha = \frac{\pi}{2} - \epsilon \qquad \cos\beta = \sin t\sin\epsilon \tag{4.14}$$

The center depends only on the energy of the photon, whereas the radius is scaled by a sinusoidal time factor. To find the particle trajectory defined by the holonomy, we can look at the intersection of w_- and w_+. This gives us

$$\tanh\chi = \sin t. \tag{4.15}$$

Comparing to the angular-independent geodesic of the form

$$\tanh\chi = \tanh\xi\sin t \tag{4.16}$$

where ξ is the rapidity of the particle, we see that the particle trajectory is that of a particle traveling with rapidity ∞, or equivalently with velocity 1, which is precisely what we expect from a photon.

There are some subtleties with regard to the single particle holonomy. The photon enters from a space-like infinity at $t = -\pi/2$, passes the origin at $t = 0$, and exits at a space-like infinity at $t = \pi/2$. The equation for the particle trajectory suggests that for $\pi/2 < t < 3\pi/2$, the photon travels backward through the space. This clearly violates the notion that our holonomy is for a particle with momentum in the $+x$ direction. The resolution to this paradox is that though the wedges oscillate after $t = \pi/2$, they actually represent a coordinate map of empty AdS space. This issue is discussed further in [6].

Turning to the the problem of a two photon collision, we can again use the approach of effective holonomies. We will look at two photons each with energy $\tan\epsilon$, one traveling in the $+x$ direction and one traveling in the $-x$ direction. As we will see, the physics of the collision depends on the value of ϵ. In particular, for sufficiently large ϵ, we will see the formation of a black hole.

Our particles have the following energy-momentum three-vectors:

$$\vec{p}_1 = (\tan\epsilon, \tan\epsilon, 0), \qquad \vec{p}_2 = (\tan\epsilon, -\tan\epsilon, 0). \tag{4.17}$$

The holonomies associated with these energy-momentum three-vectors are:

$$u_1 = \mathbf{1} + \tan\epsilon(\gamma_0 + \gamma_1), \qquad u_2 = \mathbf{1} + \tan\epsilon(\gamma_0 - \gamma_1). \tag{4.18}$$

As discussed in the previous chapter, the effective holonomy will be the multiplication of the two holonomies. We will consider both orderings of the multiplication:

$$\begin{aligned} u_{\mathit{eff}} &= u_1 u_2 \text{ or } u_2 u_1 \\ &= (1 - 2\tan^2\epsilon)\mathbf{1} + 2\tan\epsilon(\gamma_0 \pm \tan\epsilon\,\gamma_2). \end{aligned} \tag{4.19}$$

To find the effective mass and effective momentum associated with this effective holonomy, we can compare to the general form of the holonomy.

$$u_{\mathit{eff}} = \cos M\,\mathbf{1} + \frac{\sin M}{M} P^a \gamma_a \tag{4.20}$$

By direct comparison we see that:

$$\sin\frac{M}{2} = \tan\epsilon \tag{4.21}$$

$$\vec{p}_{\mathit{eff}} = \left(\frac{M}{\sqrt{1-\tan^2\epsilon}}, 0, \pm\frac{M\tan\epsilon}{\sqrt{1-\tan^2\epsilon}}\right) \tag{4.22}$$

This y momentum corresponds to a velocity of

$$v_y = \pm\tan\epsilon. \tag{4.23}$$

As in Minkowski space, the effective momentum is different depending on the ordering of the holonomy multiplication. In this case, we have a sign difference in the y velocity. This seems to be a bit unphysical in that we expect particles to have well-defined velocities. Again, this turns out to be a coordinate effect which is luckily absent when we move to the center-of-mass frame.

Looking more closely at the equation governing the effective mass, we see a potential difficulty as ϵ increases. For $0 < \epsilon < \frac{\pi}{4}$, the equation gives a well-defined mass value, but for $\frac{\pi}{4} < \epsilon < \frac{\pi}{2}$, we get an imaginary mass! Similarly, for the velocity formula, for $0 < \epsilon < \frac{\pi}{4}$, the velocity is less than 1 as would be expected for a normal particle, but for $\frac{\pi}{4} < \epsilon < \frac{\pi}{2}$, the velocity is greater than 1, suggesting tachyon formation. So not only does our particle have two different velocities, but it is traveling faster than is allowed by causality!

4.5 A Black Hole?

Clearly there is some very interesting physics going on when $\frac{\pi}{4} < \epsilon < \frac{\pi}{2}$. Our intuition is that we are forming a black hole, but we would like to have some support for this hypothesis. In the next chapter, we will transform to a coordinate system that is a black hole solution of the Einstein equation, and there will be little doubt that we are dealing with a black hole. But in cylindrical coordinates, we can still analyze some of the properties of our object to see if we have truly created a black hole.

One thing that is not obvious from the equations in the previous section, but which is obvious from the figures in [6], is that space-time ends on a singularity. By either looking at the intersection of the effective wedge edges or by using the effective momentum directly, we can find the effective particle trajectory after the collision to be

$$\tanh\chi = \sin t \tan\epsilon, \qquad \phi = \pm\frac{\pi}{2}. \tag{4.24}$$

Space-time ends when the effective particle reaches $\tanh\chi = 1$. Again, this is by no means obvious without looking at a figure, but the general idea is that for sufficiently large ϵ, the wedges are large enough to obliterate space-time. The wedges turn out to be at their maximum "size" when the effective particle reaches the space-like infinity. The coordinate time τ when this occurs is given by

$$\sin\tau = \frac{1}{\tan\epsilon}. \tag{4.25}$$

There is a singularity at both $\phi = \pi/2$ and $\phi = -\pi/2$, but because of the holonomy, they actually represent the same point in space-time.

From other studies of black holes, we know that the only space-time events that can affect the singularity are ones that have the singularity within their causal future. There is a "radius" beyond which the singularity cannot be causally reached, and we call this radius the horizon of the black hole. The horizon should be represented by backward light cones from the points

$\tanh\chi = 1, t = \tau, \phi = \pm\pi/2$:

$$\tanh\chi\cos\left(\phi \mp \frac{\pi}{2}\right) = \cos(\tau - t). \tag{4.26}$$

The horizon geodesic "ends" on the wedge edges. By the holonomy, the ends of the light cone from $\phi = \pi/2$ join with the ends of the light cone from $\phi = -\pi/2$ to form the complete horizon.

We know from classical gravity (i.e. with no Hawking radiation to evaporate the black hole) that the length of the horizon should be constant. In principle, we can use the metric to find this length. In this particular case, the calculation turns out to be a bit difficult because we have to integrate length along the geodesic defining the horizon, but the limits of integration are the points where the horizon intersects the wedge edges. To give a flavor of the calculation, we will find the horizon length at $t = \tau - \pi/2$, which is the time when the horizon forms. See [6] for more specific calculations with respect to the horizon length.

At $t = \tau - \pi/2$, the horizon geodesic from the $\phi = \pi/2$ singularity is the same as the horizon geodesic from the $\phi = -\pi/2$ singularity. This is how we know that the horizon forms at this t value. Note that the horizon lies entirely along $\phi = 0$. As mentioned above, the horizon ends on the wedge ends of the individual photons. Repeating the calculations from the last section, the upper wedge edges are given by:

$$w_1 : \tanh\chi\sin(\epsilon + \phi) = \sin t\sin\epsilon, \qquad w_2 : \tanh\chi\sin(\epsilon - \phi) = -\sin t\sin\epsilon. \tag{4.27}$$

For $t = \tau - \pi/2$, we have specifically

$$w_1 : \tanh\chi = \cos\tau, \qquad w_2 : \tanh\chi = -\cos\tau. \tag{4.28}$$

Because there is no change in ϕ or t, along the horizon, $ds = d\chi$. We know the limits of $\tanh\chi$ from the wedge edges. This suggests making the change of variables $y = \tanh\chi$.

$$dy = d\chi\,\text{sech}^2\chi = d\chi(1 - \tanh^2\chi)$$

$$d\chi = \frac{dy}{1 - y^2} \tag{4.29}$$

The total length of the horizon will be twice the distance between the wedge edges. This is because the horizon is represented by both the horizon geodesic from the upper singularity and the horizon geodesic from the lower singularity. Let μ be equal to half the horizon length. Integrating along the horizon from one wedge edge to the other:

$$\mu = \int_{-\cos\tau}^{\cos\tau}\frac{dy}{1 - y^2} = 2\,\text{arctanh}\cos\tau. \tag{4.30}$$

Through some trigonometric identities, we arrive at the result

$$\cosh\frac{\mu}{2} = \tan\epsilon. \tag{4.31}$$

What about the problem of tachyon formation? Though we can eliminate this problem by moving to the center of mass frame, we should still be able to address this issue within our current coordinate system. In particular, tachyon formation is of no concern if it occurs within the horizon of the black hole. This is because information about the tachyon formation would not be transmitted to the region outside of the blach hole. From the figures in [6], we see readily that the tachyons form well within the horizon. We refer to this phenomenon as cosmic censorship, in that causally forbidden objects such as tachyons cannot affect our causal reality outside the black hole.

Therefore, we have three reasons to believe that we are dealing with a black hole in this model: space-time ends on singularity, the length of the horizon is conserved, and we observe the cosmic censorship of tachyons. Now we would like to move to the rest frame of the black hole in order to gain further insight into the process of black hole formation. This will also eliminate the coordinate effect associated with our effective particle having two velocities.

4.6 The Black Hole Rest Frame

We would like to be able to move to a coordinate system where

$$u_{\mathit{eff}} = e^{M\gamma_0}. \tag{4.32}$$

This would correspond to an effective mass at rest. But we already saw that for $\epsilon > \frac{\pi}{4}$, M would be imaginary. Somehow we need to find an extension of the notion of mass in order to define the black hole rest frame. Consider the following holonomy:

$$u_{\mathit{eff}} = e^{-\mu\gamma_1} = \cosh\mu\,\mathbf{1} - \sinh\mu\,\gamma_1. \tag{4.33}$$

The trace of this holonomy is $\cosh\mu$, which is a function that is always greater than 1. In the case of a real mass, the trace of the holonomy was $\cos M$, which is a function that is always less than 1. In this way, μ turns out to be an analytic extension of M.

The choice of the name μ is purposeful. We will see that μ is indeed half the length of the horizon. Also, the choice of the effective holonomy is not unique, we could have chosen

$$u_{\mathit{eff}} = e^{-\mu(q^1\gamma_1 + q^2\gamma_2)} \tag{4.34}$$

as long as $(q^1)^2 + (q^2)^2 = 1$. The point is that we are looking for a holonomy that has trace greater than 1 in order to define a holonomy with an imaginary mass.

Though it is possible to Lorentz transform u_1 and u_2 to find μ, the calculation is quite tedious. A more straightforward approach is to start with μ and find photon momenta that result in the proper effective holonomy. Consider the following energy-momentum three-vectors:

$$\vec{p}_1 = (\tan\epsilon_1, -\tan\epsilon_1\cos\theta, -\tan\epsilon_1\sin\theta), \qquad \vec{p}_2 = (\tan\epsilon_2, -\tan\epsilon_2, 0). \tag{4.35}$$

This looks very similar to the center-of-mass particle momenta in flat space in that we have one photon coming in from the x direction and one photon coming in at some angle from the x axis. The associated holonomies are:

$$u_1 = \mathbf{1} + \tan\epsilon_1(\gamma_0 - \cos\theta\gamma_1 - \cos\theta\gamma_2), \qquad u_1 = \mathbf{1} + \tan\epsilon_2(\gamma_0 - \gamma_1). \tag{4.36}$$

We are looking for a solution to:

$$u_1 u_2 = e^{-\mu\gamma_1}. \tag{4.37}$$

Expanding and solving this equation, we find

$$\begin{aligned} \tan\epsilon_1 &= \coth\frac{\mu}{2}\cosh\mu, \\ \tan\epsilon_2 &= \coth\frac{\mu}{2}, \\ \sin\theta &= \tanh\mu. \end{aligned} \tag{4.38}$$

Thus, the problem is completely determined in terms of μ, and this properly defines the black hole rest frame.

Because we are interested in the geodesic distance between the photons, we should know the wedge edge that connects the two photons. Each photon has two wedge edges, but they share one of these edges in common. In order to find the geodesic distance between the photons, we need to identify this common edge. Then we can either look at the wedge equation from the first photon or the wedge equation from the second photon because they should be identical. The expression for ϵ_2 is slightly simpler than the expression for ϵ_1, so we will focus on the wedge edge from the second photon.

$$\tanh\chi\sin(\epsilon + \phi) = -\sin t \sin\epsilon_2. \tag{4.39}$$

We can check that this wedge edge runs from the second photon at $\phi = 0$ to the first photon at $\phi = \theta$.

Now we have the beginnings of a model of black hole formation. We have photon trajectories in cylindrical coordinates which have an effective holonomy with imaginary mass. This imaginary mass is related to the length of the black hole horizon. Because we have identified the geodesic that connects the two photons, we can study the relative dynamics. This will allow us to write down an expression for the relative Hamiltonian near the effective horizon.

Chapter 5

Black Hole Formation in AdS Space

5.1 Toward a Relative Hamiltonian

In the previous chapter, we saw how the collision of two photons in AdS space could create a black hole with a well-defined horizon. We moved to the rest frame of the black hole and were able to find the appropriate photon trajectories. In this chapter, we will study the relative motion of these photons in order to come up with a Hamilonian that reproduces the relative motion near the horizon of the black hole. With this information, we will be able to calculate $\overline{S}$, telling us whether or not our black hole formation process is semi-classically suppressed or not.

The key is geodesics. Though we will not be able to find a relative coordinate and momentum pair, we will be able to use the length of the geodesic that connects the photons to tease out an expression for the relative Hamiltonian. However, the relative Hamiltonian in cylindrical coordinates would not be particularly interesting, because there are no effects near the horizon. Thus, we will move to BTZ black hole coordinates before finding a relative Hamiltonian in order to see the horizon effects.

5.2 Geodesic Distance in Cylindrical Coodinates

First, let us look at the geodesic distance between the photon in the rest frame of the black hole. In terms of the spatial coordinates χ and ϕ and the time coordinate t, we saw that the photon trajectories were the following:

$$\begin{array}{llll} \textit{Particle 1:} & \tanh\chi = \sin t & \phi = 0 & \\ \textit{Particle 2:} & \tanh\chi = \sin t & \phi = \theta & \sin\theta = \tanh\mu \end{array} \tag{5.1}$$

Recall that μ is a parameter that measures the half that horizon length of the black hole. The energies of the particles are parameterized as

$$\begin{aligned} E_1 &= \tan\epsilon_1 = \coth\frac{\mu}{2}\cosh\mu, \\ E_2 &= \tan\epsilon_2 = \coth\frac{\mu}{2}. \end{aligned} \tag{5.2}$$

The geodesic that connects the points is described by

$$\tanh\chi\sin(\epsilon+\phi) = -\sin t\sin\epsilon_2, \tag{5.3}$$

and this geodesic runs from $\phi = 0$ to $\phi = \theta$.

The metric for the spatial part of AdS space in cylindrical coordinates is

$$ds^2 = d\chi^2 + \sinh^2\chi\, d\phi^2. \tag{5.4}$$

Because the part of the geodesic we are interested in runs from well-defined ϕ values, it is convenient to parametrize the geodesic in terms of ϕ. Thus, the geodesic length will be

$$S(t) = \int_0^\theta d\phi\,\sqrt{\left(\frac{d\chi}{d\phi}\right)^2 + \sinh^2\chi}. \tag{5.5}$$

To find $\frac{d\chi}{d\phi}$ we can differentiate the equation describing the geodesic implicitly with respect to ϕ.

$$\begin{aligned} \tanh\chi\cos(\epsilon+\phi) + \frac{1}{\cosh^2\chi}\frac{d\chi}{d\phi}\sin(\epsilon+\phi) = 0, \\ \frac{d\chi}{d\phi} = \frac{-\sinh\chi\cosh\chi}{\tan(\epsilon+\phi)}. \end{aligned} \tag{5.6}$$

Substituting this into the integral and using the equation describing the geodesic and some trigonometric identities, we find that

$$S(t) = \int_0^\theta d\phi\,\frac{\sin t\sin\epsilon_2\sqrt{1-\sin^2 t\sin^2\epsilon_2}}{\sin^2(\epsilon_2+\phi) - \sin^2 t\sin^2\epsilon_2}. \tag{5.7}$$

Because ϵ_2 and θ are described in terms of μ, this geodesic length will be given in terms of t and μ alone. Evaluating the integral, we find the following geodesic distance:

$$S(t) = -2\ \operatorname{arctanh}\left(\frac{\sin t}{\sqrt{1+\cos^2 t\coth\mu/2}}\right). \tag{5.8}$$

Time runs from $t = -\pi/2$ to $t = 0$ and we can see from Figure 5.1 that as t goes to 0, $S(t)$ smoothly approaches 0. Unfortunately, this does not look like black hole formation. As we saw in the introductory chapter, we expect the velocity to approach zero near the horizon. We need to find a coordinate system that exhibits this property.

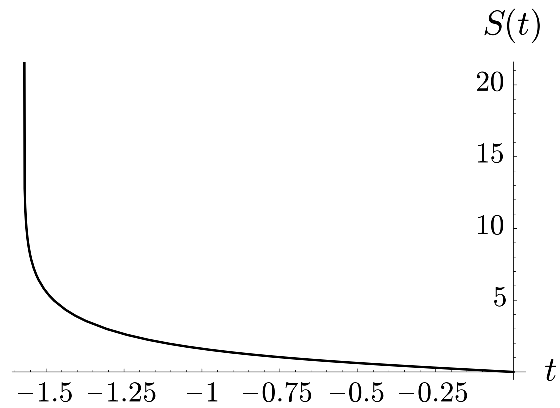

Figure 5.1: Graph of $S(t)$ for $\coth \mu/2 = 2$.

5.3 From Cylindrical Coordinates to the BTZ Black Hole

The BTZ black hole is a solution to the Einstein equations with a mass source at the origin and negative cosmological constant. With the holonomy defined by $e^{-\mu\gamma_1}$, cylindrical AdS space is exactly mapable to BTZ coordinates.

We would like to perform the change to the BTZ coordinates, because in the rest frame of the black hole, we do not see any special events near the horizon. But in the BTZ coordinate system, the formation of the horizon can be explicitly seen. The metric for the usual BTZ black hole is

$$ds^2 = -\left(\frac{\tilde{r}^2}{\ell^2} - 8GM\right) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\frac{\tilde{r}^2}{\ell^2} - 8GM} + \tilde{r}^2 d\tilde{\phi}^2, \tag{5.9}$$

with the tilde introduced to avoid confusion with our cylindrical coordinates. A horizon is formed when there is a singularity in the radial part of the metric and a zero in the time part of the metric. This occurs at $\tilde{r} = \mu = \ell\sqrt{8GM}$. (This M should not be confused with the M in the previous chapters.) In fact, the length of the horizon is 2μ, because for fixed r and t, $ds = \tilde{r}d\tilde{\phi}$ and $\tilde{\phi}$ runs from -1 to 1.

To make the algebra a bit simpler, consider a change from the coordinates $\tilde{r}$, $\tilde{\phi}$, and $\tilde{t}$, to new coordinates R, Φ, and T given by

$$R = \frac{\tilde{r}}{\mu}, \qquad \Phi = \mu\tilde{\phi}, \qquad T = \mu\left(\tilde{t} + \frac{\pi}{2}\right). \tag{5.10}$$

Looking only at the region outside of the horizon $\tilde{r} = \mu$, the range of these coordinates are

$$1 < R < \infty, \qquad -\mu < \Phi < \mu, \qquad \frac{\mu\pi}{2} < T < \infty. \tag{5.11}$$

The metric is now

$$ds^2 = -(R^2-1)dT^2 + \frac{dR^2}{R^2-1} + R^2 d\Phi^2. \tag{5.12}$$

Note that the horizon length is still 2μ, because the horizon forms at $R = 1$ and Φ runs from $-\mu$ to μ.

In these coordinates we can see quite clearly the connection between the BTZ black hole and AdS space itself. In fact, in terms of the x_i variables we introduced in chapter 4:

$$\begin{aligned} x_{-1} &= \sqrt{R^2-1}\sinh T \\ x_0 &= R\cosh\Phi \\ x_1 &= \sqrt{R^2-1}\cosh T \\ x_2 &= R\sinh\Phi \end{aligned} \tag{5.13}$$

As a consequence, one has a relationship between cylindrical AdS coordinates and BTZ coordinates. Applying these transformations, we can find the trajectories of the particles in the new frame. The first particle trajectory was given by $\tanh\chi = \sin t$ and $\varphi = 0$. Dividing the expression for x_2 by the expression for x_0 and comparing to the equations for the x_i variables in cylindrical coordinates:

$$\tanh\Phi = \frac{\tanh\chi\sin\varphi}{\sin t} = 0, \tag{5.14}$$

so $\Phi = 0$. A similar calculation for the second particle shows that $\Phi = \mu$. To find R as a function of T, we can divide the the expression for x_1 by the expression for x_0:

$$\frac{\sqrt{R^2-1}\cosh T}{R\cosh\Phi} = \frac{\tanh\chi\cos\varphi}{\sin t}. \tag{5.15}$$

For both particles, $\cosh\Phi\cos\varphi = 1$, so

$$\begin{aligned} \sqrt{R^2-1}\cosh T &= R, \\ R &= \coth T. \end{aligned} \tag{5.16}$$

To summarize:

$$\begin{array}{lll} \textit{Particle 1:} & R = \coth T & \Phi = 0 \\ \textit{Particle 2:} & R = \coth T & \Phi = \mu \end{array} \tag{5.17}$$

What about the geodesic in the BTZ coordinates? Taking the equations describing the geodesic in the cylindrical coordinates, expanding the $\sin(\epsilon + \varphi)$ term and multiplying by $\cosh\chi$ we have

$$\sinh\chi\sinh\phi\cos\epsilon + \sinh\chi\cos\varphi\sin\epsilon = -\cosh\chi\sin t\sin\epsilon. \tag{5.18}$$

Using the coordinate transformations and letting $\sin\epsilon = \coth\mu/2 = w$, we arrive at

$$R\sinh\Phi + w\sqrt{R^2-1}\cosh T = -wR\cosh\Phi,$$

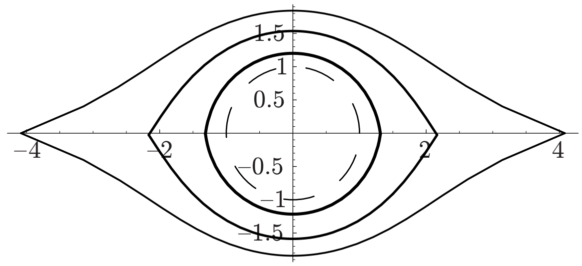

Figure 5.2: Graph of BTZ geodesic for $w = 2$, and $T = .25, .50, 1.00$.

$$R^2 = \frac{w^2 \cosh^2 T}{w^2 \cosh^2 T - (\sinh \Phi + w \cosh \Phi)^2}. \tag{5.19}$$

One might wonder what the BTZ geodesic looks like in the BTZ coordinates. Figure 5.2 is a graph of the geodesic at three different times T. The plot is of R as a radial coordinate and Φ as an angular coordinate. The dashed line represents the horizon of the black hole.

Note that the inside of each "football" is an excised region of space-time. We know that in cylindrical coordinates, the two photons cut out a region from space-time in addition to the region defined by the holonomy $e^{-\mu\gamma_1}$. This additional excised regions falls between the wedge edges that connect the photons, therefore the space within the footballs must be subtracted and point along the football are identified with symmetric points across the x axis.

As T goes to infinity, $R(\Phi)$ approaches but does not cross the horizon of the black hole. In cylindrical coordinates, the particles enter the horizon and the distance between them approaches zero. At first glance, this would seem to indicate two different physical realities, but this is actually a general feature of black hole physics: when coordinate systems exchange time and space coordinates, there are often strange phenomena at the horizon.

We have two remaining calculations before we can discuss relative motion and relative Hamiltonians. First, we want to make sure that the geodesic from cylindrical coordinates is still a geodesic in the BTZ coordinates. Once we have verified this, we want to find the BTZ geodesic distance as a function of T.

5.4 Verifying the Transformed Geodesic

The reason why this calculation is necessary is that in changing from cylindrical coordinates to BTZ coordinates, we have mixed the space and time coordinates. Thus, looking at a curve at a fixed time T is very different from looking at a curve at a fixed time t. It turns out, however, that the transformed geodesic is still a geodesic in the BTZ coordinates.

First, we need to calculate the Christoffel symbols for the BTZ metric. The spatial BTZ metric is:

$$ds^2 = \frac{dR^2}{R^2 - 1} + R^2 d\Phi^2,$$

$$g_{11} = \frac{1}{R^2 - 1}, \qquad g_{22} = R^2, \qquad g_{12} = g_{21} = 0. \tag{5.20}$$

Calculating some relevant inverses and derivatives:

$$g^{11} = R^2 - 1, \qquad g^{22} = \frac{1}{R^2},$$

$$g_{11,1} = \frac{-2R}{(R^2 - 1)^2}, \qquad g_{22,1} = 2R. \tag{5.21}$$

The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell}). \tag{5.22}$$

The nonvanishing Christoffel symbols are

$$\Gamma^1_{11} = \frac{-R}{R^2 - 1}, \qquad \Gamma^2_{12} = \frac{1}{R}, \qquad \Gamma^1_{22} = -R(R^2 - 1). \tag{5.23}$$

Given coordinates x^i and an arc length parameter s, the geodesic equation is

$$\frac{d^2}{ds^2} x^i + \Gamma^i_{jk} \frac{d}{ds} x^j \frac{d}{ds} x^k = 0. \tag{5.24}$$

For the BTZ metric, and letting dots signify derivatives with respect to the arc length parameter, this yields two equations:

$$\ddot{\Phi} + \frac{2}{R} \dot{R} \dot{\Phi} = 0, \tag{5.25}$$

$$\ddot{R} - \frac{R}{R^2 - 1} \dot{R}^2 - R(R^2 - 1)\dot{\Phi}^2 = 0. \tag{5.26}$$

Because our candidate geodesic is written as $R(\Phi)$, we can exchange derivatives with respect to s (dots) for derivatives with respect to ϕ (primes):

$$\dot{R} = R' \dot{\Phi}, \qquad \ddot{R} = R'' \dot{\Phi} + R' \ddot{\Phi}. \tag{5.27}$$

Plugging these into the geodesic equations and eliminating $\ddot{\Phi}$ and $\dot{\Phi}^2$ dependence, we arrive at

$$RR'' + \frac{2 - 3R^2}{R^2 - 1} R'^2 - R^2(R^2 - 1) = 0. \tag{5.28}$$

It looks like this equation might be simpler in terms of $L = R^2$.

$$L' = 2RR', \qquad R'^2 = \frac{L'^2}{4L},$$

$$L'' = 2RR'' + 2R'^2, \qquad RR'' = \frac{L''}{2} - \frac{L'^2}{4L}. \tag{5.29}$$

The geodesic equation is now

$$\frac{L''}{2} + \frac{3-4L}{4L(L-1)}L'^2 - L(L-1) = 0. \tag{5.30}$$

Finally, our candidate geodesic is of the form

$$L = \frac{A}{A - f^2}, \tag{5.31}$$

where A is a constant and f is function of Φ. With this substitution

$$L' = \frac{2Aff'}{(A-f^2)^2},$$

$$L'' = \frac{2Af'^2}{(A-f^2)^2} + \frac{2Aff''}{(A-f^2)^2} + \frac{8Af^2f'^2}{(A-f^2)^3}, \tag{5.32}$$

and after some simplification, we arrive at

$$f = f''. \tag{5.33}$$

For the candidate geodesic, $f = \sinh\Phi + w\cosh\Phi$, which indeed satisfies this condition.

5.5 Geodesic Distance in the BTZ Metric

We are now ready to find the geodesic distance in the BTZ metric. In analogy with the case of cylindrical coordinates, it is convenient to parametrize in terms of the variable Φ. The geodesic distance is given by

$$D(T) = \int_0^\mu d\Phi \sqrt{\frac{1}{R^2-1}\left(\frac{dR}{d\Phi}\right)^2 + R^2}. \tag{5.34}$$

Using implicit differentiation on the equation that defines the BTZ geodesic, we find that

$$\frac{dR}{d\Phi} = -R^2\sqrt{R^2-1}\,\frac{w\sinh\Phi + \cosh\Phi}{w\cosh T}. \tag{5.35}$$

Eliminating R in favor of Φ using various trigonometric identities, our integral becomes

$$D(T) = \int_0^\mu d\Phi\, \frac{w\cosh T\sqrt{1+w^2\sinh^2 T}}{w^2\cosh^2 T - (\sinh\Phi + w\cosh\Phi)^2}. \tag{5.36}$$

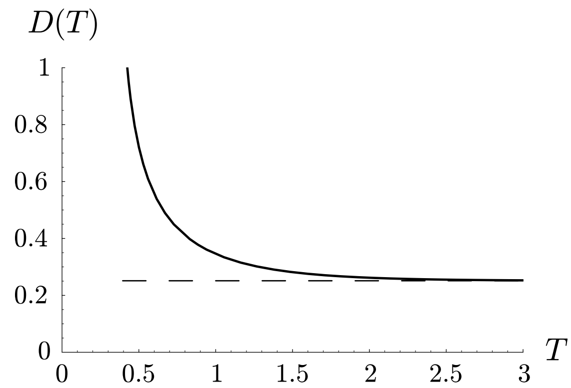

Figure 5.3: Graph of $D(T)$ for $w = \coth \mu/2 = 8$.

Because w is a function of μ, this geodesic equation only depends on T and μ. Evaluating the integral, we find that:

$$D(T) = \operatorname{arctanh}\left(\frac{3 + w^2 + 2w^2 \cosh^2 T}{(1 + w^2)\cosh T \sqrt{1 + w^2 \sinh^2 T}}\right) - \operatorname{arctanh}\left(\frac{1}{\cosh T \sqrt{1 + w^2 \sinh^2 T}}\right). \quad (5.37)$$

As we can see from Figure 5.3, as T goes to infinity, $D(T)$ approaches the limiting value of μ. Thus $\dot{D} \to 0$ near the horizon, which is precisely what we expect for black hole formation.

5.6 Relating Relative Distance and Velocity

We are now ready to find an expression for the relative Hamiltonian. Given a Hamiltonian, the equation of motion is given by

$$\dot{D} = \frac{\partial H}{\partial P_D}.$$

We would like to find a Hamiltonian that gives the appropriate expression for $\dot{D}$ in the limit of $D \to \mu$ or equivalently $T \to \infty$. To do this, we need to find a relation between D and $\dot{D}$ in this limit.

We can expand $D(T)$ in powers of e^T about the point $T = \infty$.

$$D(T) = \operatorname{arctanh}\left(\frac{2w}{1 + w^2}\right) + \left(\frac{4(1 + 6w^2 + w^4)}{w(w^2 - 1)^2}\right) e^{-2T} + O(e^{-4T}). \quad (5.38)$$

Recalling that $w = \coth \mu/2$, we have

$$D(T) \approx \mu + 4 \cosh 2\mu \tanh \frac{\mu}{2} e^{-2T}, \quad (5.39)$$

and thus

$$\dot{D}(T) \approx -8\cosh 2\mu \tanh\frac{\mu}{2} e^{-2T}. \tag{5.40}$$

Clearly, as $T \to \infty$, we have the relation

$$\dot{D} = 2(\mu - D), \tag{5.41}$$

or dropping terms of order e^{-4T},

$$\dot{D} = \frac{\mu^2 - D^2}{\mu}. \tag{5.42}$$

This second relation looks exactly like a particle falling in a black hole background that we saw in the first chapter. However, because of the connection between μ and energy, the Hamiltonian will be very different.

5.7 The Approximate Hamiltonian

From our experience with black holes, we know that the mass of the effective black hole is the total energy of the system. For the BTZ black hole, $\mu = \ell\sqrt{8GM}$ so in units where $8G = \ell = 1$, the Hamiltonian $H = \mu^2$. It is the fact that in the collision case, the black hole mass is given by the Hamiltonian itself which leads to a difference from the situation in the case of a test particle falling in a black hole background. Now the horizon radius is a dynamical variable related to the Hamiltonian, thus the Hamiltonian must be self-consistently determined.

We can reproduce the equation relating D and $\dot{D}$ by taking a Hamiltonian of the form

$$H = D^2 \tanh^2\frac{P}{2D}. \tag{5.43}$$

In the limit $D^2 \to \mu^2 = H$, this Hamiltonian will give us the desired relation for $\dot{D}$.

$$\begin{aligned}\frac{\partial H}{\partial P} &= 2D^2 \tanh\frac{P}{2D}\operatorname{sech}^2\frac{P}{2D}\frac{1}{2D} = D\tanh\frac{P}{2D}\left(1 - \tanh^2\frac{P}{2D}\right)\\ &= D\left(\frac{-\sqrt{H}}{D}\right)\left(1 - \frac{H}{D^2}\right) = \frac{\sqrt{H}}{D^2}(H - D^2)\\ &\approx \frac{H - D^2}{\sqrt{H}} = \frac{\mu^2 - D^2}{\mu}.\end{aligned} \tag{5.44}$$

Note that when substituting for $\tanh\frac{P}{2D}$, we use the negative square root of the Hamiltonian. This is because when $D^2 \to H$, $P \to -\infty$. In any case, we have arrived at the expected relation.

We can look more closely the momentum behavior near the horizon. For large momenta, we can expand the hyperbolic tangent.

$$H - D^2 = -4D^2\, e^{\frac{P}{2D}} \tag{5.45}$$

We see that for distances $D \sim H^{1/2}$, P does approach $-\infty$. More explicity, we have

$$P \approx 2D \ln \frac{D^2 - H}{4D^2}. \tag{5.46}$$

Compared with the particle case, where the momentum was diverging linearly, we now have a milder logarithmic divergence. This will change the behavior of $\overline{S}$.

5.8 Calculating $\overline{S}$

Now that we have an expression for the relative Hamiltonian, we can evaluate the action for the classical process of black hole formation. As we mentioned in the first chapter, we are interested in the quantity

$$\overline{S} = \int^{\mu} P dD. \tag{5.47}$$

Let us evaluate this quantity for our approximate Hamiltonian. The momentum is given by

$$P = -2D \operatorname{arctanh} \frac{\mu}{D}. \tag{5.48}$$

The following change of variables will be helpful:

$$z = \frac{\mu}{D}, \qquad dz = -dD \frac{\mu}{D^2}, \qquad dD = -dz \frac{\mu}{z^2}. \tag{5.49}$$

Evaluating the integral for $\overline{S}$:

$$\int^{\mu} P dD = 2\mu^2 \int^{1} \frac{dz}{z^3} \operatorname{arctanh} z = 2\mu^2 \left(\frac{1}{2} \left(1 - \frac{1}{z^2} \right) \operatorname{arctanh} z - \frac{1}{2z} \right) \Bigg|^{1} = -\mu^2. \tag{5.50}$$

Here, we have used the limit

$$\lim_{z \to 1} \left(1 - \frac{1}{z^2} \right) \operatorname{arctanh} z = 0. \tag{5.51}$$

The important result is that $\overline{S}$ is finite. Thus, we see that the action for the classical process is finite, and therefore it cannot have a imaginary contribution. As a consequence, apart from the energy divergence, the semi-classical amplitude is a pure phase.

$$A \propto e^{\frac{i}{\hbar}\overline{S}} = e^{-\frac{i}{\hbar}\mu^2}, \qquad |A|^2 = 1. \tag{5.52}$$

This implies that at the semi-classical level, there is no exponential suppression or enhancement of the process of black hole formation.

5.9 Summary and Conclusions

We began this paper with the goal of trying to find a model of black hole formation that would allow us to understand whether black hole formation is semi-classically suppressed or not. Along the way, we developed the language of gravity in $2+1$ dimensions, allowing us to use simple algebra and geometry to understand the relative dynamics of point particles. This led to a model of black hole formation based on the head-on collision of photons in a space with negative cosmological constant. Once we found a Hamiltonian that described the near horizon behavior, we were able to calculate $\overline{S}$ to show that there was no possibility for exponential suppression.

Ideally, we would like to repeat this calculation in $3+1$ dimensions. Unfortunately, as mentioned in the introduction, there is no analytic solution to the two-body problem in $3+1$ dimensions. In fact, the reason why previous models of black hole formation were based on a test particle falling in an effective black hole background is that this seemed like the only way to begin talking about black hole formation. As this paper shows, however, when we take into account the exact equations of motion for the relative dynamics, the momentum divergence changes from linear to logarithmic. This occurred because the black hole horizon length was directly related to the Hamiltonian of the system, so we had to find a self-consistent form of the Hamiltonian in order to discuss the relative dynamics.

Recently, an exactly solvable model of black hole formation in $3+1$ dimensions was presented based on point particles with associated gravitational waves [8]. The advantage of this approach is that it readily generalizes to higher dimensions, though the existence of analytic solution above $3+1$ dimensions has not been established. Because this model demonstrates the existence of a horizon, it should be possible to perform the same kind of semi-classical analysis as presented in this paper. Our hope is that the switch from linear divergence of the momentum for test particles (which is true in all dimensions) to logarithmic divergence for pairs of particles (which we have only showed in $2+1$ dimensions) will be shown to hold universally.

At the very least, this paper lends credence to the idea that black hole formation is a purely geometric phenomena. That is to say, once the distance between two particles is smaller than the effective horizon, a black hole will form without suppression. Thus, we have reason to trust studies that try to estimate the probability of black hole formation in particle accelerators by looking at a geometrical cross-sections. Also, strings in $3+1$ dimensions can behave like point particles in $2+1$ dimensions if there is sufficient symmetry, so the model presented in this paper could be the basis for an analysis of "black string" formation.

In this paper, we have offered a counter-example to the theory of exponential suppression. By showing that the momentum divergence behavior depends on the model chosen, we emphasize the need to understand the limitations of different models of black hole formation. Our model is exactly solvable, but our universe is certainly not $2+1$ dimensional. Which of these factors will be more important depends on future experimental and theoretical work.

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For a more complete bibliography that covers articles dealing with various other aspects of black hole formation, see A. Jevicki and J. Thaler, *Dynamics of Black Hole Formation in an Exactly Solvable Model*, hep-th/0203172.