Gravity and Unitarity

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1 Motivation

In the hearts and minds of physicists, gravity holds a mystical appeal. For the general relativist, Einstein's principle of equivalence is the foundation for a universe of metric manifolds. For the field theorist, gravity is a highly coupled theory that begs for a UV completion. For the string theorist, gravity is a closed string. (Oh, the perfection of an *enso*, so Zen!) And for every other physicist, gravity is the ultimate weak force, completely ignorable save a few factors of g in Physics 15a.

But as a low energy effective theory, gravity is merely a massless spin-2 field, whose interactions with matter (and itself) are completely determined by Lorentz invariance and unitarity. In GR, we often think of replacing Lorentz invariance (*i.e.* flat space symmetry) with general covariance (*i.e.* arbitrary curved space symmetry), but in the language of field theory we should really make the following analogy:

$$\text{electromagnetism : gravity :: gauge symmetry : general covariance} \tag{1}$$

In other words, the fact that gravity is a geometric theory will not at all be obvious when we describe it as a spin-2 field, and general covariance will not be a guiding principle in forming our theory but a consequence of unitarity.

The issue gets even more confused when we talk about doing field theory in curved space-times. There, gravity *is* the geometry of space-time and field theory is just a perturbation on top of this background geometry. In the effective field theory language, we will "linearize" gravity, which at first glance seems to treat gravity as just a perturbation on field theory. But our EFT will have general covariance to *all* orders in perturbation theory, and unitarity will demand gravity to be described by Einstein's Lagrangian. This is no longer just linearized gravity; it's the real thing! And just like the real thing, we expect the theory to break down near the Planck scale.

We have all been taught to think of gravity as being fundamentally different than the other forces. And while this is technically true (gravity is associated with a spin-2 field whereas all the other forces are associated with spin-1 fields), it is also illuminating to see how gravity can be incorporated into the effective field theory language. Gravity will no longer be a manifestly geometric theory (though all the geometric predictions of time-dilation, etc. would still appear, albeit in a different language), but what we lose in geometric intuition, we gain in seeing that gravity is the (almost) unique effective theory of a massless spin-2 field.

2 The Graviton

We often think of the graviton as being related to the metric tensor $g_{\mu\nu}$. This isn't exactly true. If we define:

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + 2\lambda h_{\mu\nu} \tag{2}$$

then $h_{\mu\nu}$ is the massless spin-2 particle we call the graviton and $\lambda = \sqrt{8\pi G}$ is the strength of the gravitational interaction. If we take the generally covariant matter action of a Klein-Gordon field and expand to first order in λ:

$$\begin{aligned} \mathcal{S}_{matter} &= \int d^4x \sqrt{g}\, \frac{1}{2}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2) \\ &= \int d^4x \, \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) - \lambda\int d^4x \left(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}h\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}hm^2\phi^2\right) \end{aligned} \tag{3}$$

where now we raise and lower the indices as Lorentz indices and $h = h^\mu_\mu$. In this linearized form, we see the standard kinetic and mass terms for ϕ as well as (almost) all Lorentz invariant combinations involving just one $h_{\mu\nu}$. I say almost, because we don't have an $h\phi^3$ term which would be marginal, and so on.

Note, however, that the coefficients of the terms we do have are set by general covariance. Our hope is that these coefficients will also be set by the requirement of unitarity. Similarly, we want to figure out what unitarity says about quadratic terms (*i.e.* the propagator) for $h_{\mu\nu}$. Before doing that, let's review how these arguments worked in Abelian gauge theories.

3 Revisiting Gauge Theories

Sourceless, classical electromagnetism is described by Maxwell's equation,

$$\partial_\mu F^{\mu\nu} = 0. \tag{4}$$

If we expand A_μ in plane waves

$$A_\mu(x) = \epsilon_\mu(p)e^{ipx}, \tag{5}$$

then in momentum space, Maxwell's equation reads

$$p_\mu(p^\mu\epsilon^\nu - p^\nu\epsilon^\mu) = 0, \tag{6}$$

where ϵ is called the polarization vector. Putting the photon on-shell ($p^2 = 0$), this equation reduces to

$$p \cdot \epsilon = 0. \tag{7}$$

Note that $\epsilon \propto p$ is a valid albeit content-free solution to Maxwell's equation. That's because Maxwell's equation (on shell) is invariant under

$$\epsilon_\mu \to \epsilon_\mu + \alpha p_\mu. \tag{8}$$

We know that if we have a physical process involving external photons, the physical amplitude is given by

$$\mathcal{M} = \epsilon_\mu(p)\cdots\epsilon_\nu(q)^*\cdots\mathcal{M}^{\mu\nu\cdots}(p, q, \cdots), \tag{9}$$

where $\epsilon_\mu(p)$ is the polarization of an incoming photon $\epsilon_\nu(q)^*$ is the polarization of an outgoing photon and $\mathcal{M}^{\mu\nu\cdots}$ is the amputated Feynman diagram. We want this physical amplitude to be Lorentz invariant, so let's figure out how the polarization vectors transform under a Lorentz boost. Naming the two polarizations $\epsilon_\pm$:

$$\epsilon_+^\mu \rightarrow \alpha_+\epsilon_+^\mu + \alpha_-\epsilon_-^\mu + \alpha_k k^\mu \tag{10}$$

for some α values related to Λ_μ^ν. (Note that under a Lorentz boost, $p\cdot\epsilon$ still equals zero so we don't have to worry about the fourth direction.) This poses a bit of a problem because k^μ is not a physical polarization. In order for our physical amplitude to be a Lorentz scalar while still having physical measurements independent of the non-existent longitudinal mode, we must have

$$k_\mu \mathcal{M}^{\mu\cdots}(k,\cdots) = 0 \tag{11}$$

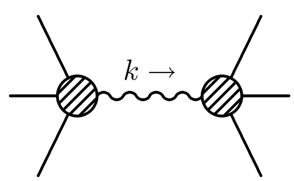
for all external photons and any number of other external particles.

For a general Lagrangian, we have no reason to expect this cancellation. Thus, we expect a symmetry to enforce this condition. We already saw that Maxwell's equation was invariant under $\epsilon_\mu \rightarrow \epsilon_\mu + \alpha p_\mu$. In position space, this becomes

$$A_\mu \rightarrow A_\mu + \partial_\mu\theta \tag{12}$$

which is precisely a gauge transformation. In some sense, once we know that there is such a symmetry, we need only build a Lagrangian that has that symmetry (plus a symmetry breaking term that we'll come to momentarily). If we weren't this smart (or if we were dealing with a highly coupled theory like a non-Abelian gauge theory and didn't know about BRST symmetry), we would simply write down a general Lorentz invariant Lagrangian, then choose the coupling constants such that $k_\mu\mathcal{M}^\mu = 0$ for every physical process we could think of. We'll come to this in a moment when coupling photons to matter.

What about the condition of unitarity? The unitarity of the S-matrix is equivalent to the optical theorem, which relates the imaginary part of the scattering amplitude from state a to a to the sum over all possible decay modes of a state a. With Feynman diagrams, the easiest way to understand the optical theorem is through the Cutkosky rules. Take a diagram like



(13)

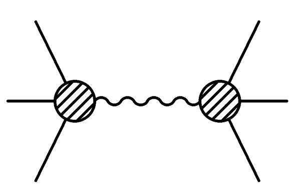
If the propagator for the photon is

$$\frac{i}{k^2 + i\epsilon}\cdot(\text{L.I. factor})_{\mu\nu} \tag{14}$$

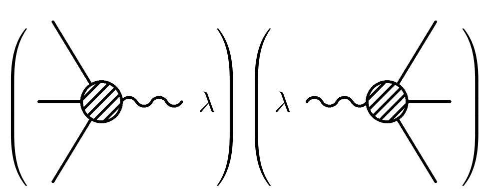
then the amplitude for this process will have an imaginary part for $k^2 = 0$, and that imaginary part will be proportional to

$$\delta(k^2)\mathcal{M}^\mu(\text{L.I. factor})_{\mu\nu}\mathcal{M}^{\nu *}, \tag{15}$$

where $\mathcal{M}^\mu$ is the amplitude represented by the blob. From the cutting rules:



$= \sum_\lambda \int d\Pi_\lambda$



(16)

Here λ represents both the polarization and possible (on-shell) momenta of the intermediate photon. Note that the right hand side of this equation is proportional to

$$\delta(k^2)\mathcal{M}^\mu \left(\sum_\lambda \epsilon_\mu^{\lambda *}\epsilon_\nu^\lambda\right)\mathcal{M}^{\nu *} \tag{17}$$

(The delta function comes from the fact that the photon is now on shell.) The equality of the left and right hand sides of the cutting equation tell us that in order to figure out the Lorentz invariant factor for the photon propagator, we need to do a sum over physical polarizations of the external photons.

If we choose an external momentum

$$k^\mu = (E, 0, 0, E), \tag{18}$$

then the physical polarizations are

$$\epsilon_\mu^1 = \frac{1}{\sqrt{2}}(0, 1, i, 0), \qquad \epsilon_\mu^2 = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \tag{19}$$

where we have used the normalization $\epsilon_\mu^{a*}\epsilon^{b\mu} = \delta^{ab}$. If we introduce

$$\tilde{k}^\mu = (E, 0, 0, -E) \tag{20}$$

then we find

$$\sum_\lambda \epsilon_\mu^{\lambda *}\epsilon_\nu^\lambda = -\eta_{\mu\nu} + \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2}. \tag{21}$$

At first glance, this seems worrisome because this is not a Lorentz invariant quantity. But already we saw that Lorentz invariance required that $k_\mu \mathcal{M}^\mu(k) = 0$, so

$$\mathcal{M}^\mu \left(\sum_\lambda \epsilon_\mu^{\lambda *}\epsilon_\nu^\lambda\right)\mathcal{M}^{\nu *} = \mathcal{M}^\mu \left(-\eta_{\mu\nu} + \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k^2}\right)\mathcal{M}^{\nu *} = \mathcal{M}^\mu \left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\mathcal{M}^{\nu *} \tag{22}$$

Thus, a photon propagator consistent with the cutting rules is

μ ν

$$= \frac{i}{k^2 + i\epsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{23}$$

for *any* fixed value of ξ. In terms of the quadratic piece of the Lagrangian, this propagator is

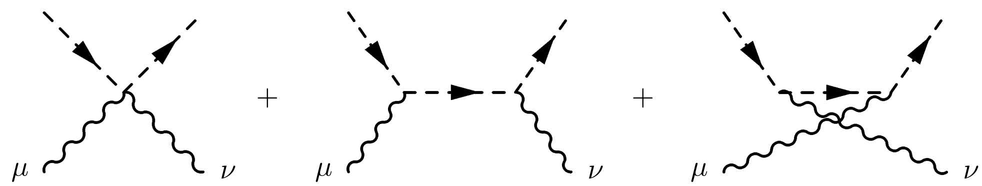
$$\mathcal{L}_{free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{24}$$

Now we see that ξ is related to the gauge fixing term, and the arbitrariness of ξ is just a statement that we can do calculations in any gauge we want, as long as we do indeed fix a gauge.

Now that we have a propagator, we know the quadratic parts for A_μ in the Lagrangian. What about interactions? Whatever interactions we put in have to satisfy the condition that $k_\mu \mathcal{M}^\mu(k) = 0$ for external photons. Let's think of the possible relevant and marginal interactions for a complex scalar with our photon field.

$$\mathcal{L}_{int} = ie\phi^*\partial^\mu\phi A_\mu + h.c. + g\phi^*\phi A_\mu A^\mu \tag{25}$$

Consider the Compton scattering of a scalar.



(26)

If we calculate $k_\mu \mathcal{M}^{\mu\nu}(k, \cdots)$ for these diagrams, we find that it is proportional to $e^2 - g$, so for our theory to be Lorentz invariant, we must have $g = e^2$. Writing out the full Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{27}$$

And now we notice the gauge symmetry (apart from the gauge fixing term)

$$A_\mu \to A_\mu + \partial_\mu \theta, \qquad \phi \to e^{ie\theta}\phi. \tag{28}$$

If we add scalar interactions like

$$\mathcal{L}_{scalar} = m^2\phi^*\phi + \mu^2(\phi\phi + \phi^*\phi^*) + h(\phi\phi\phi^* + \phi^*\phi^*\phi) + \cdots \tag{29}$$

we will find that unless $\mu = h = 0$ (*i.e.* the interactions are gauge invariant), we won't have $k_\mu \mathcal{M}^\mu(k, \cdots) = 0$ for diagrams with external photons. Thus, for massless spin-1 particles, gauge symmetry helps resolve the tension between unitarity and Lorentz invariance.

We will repeat the same construction for gravity. Using the physical polarizations, we will find the form of the graviton propagator. Then we will use the equivalent of $k_\mu \mathcal{M}^\mu(k, \cdots) = 0$ to figure out the relationships between various coupling constants. Of course, we know that gravity is a highly coupled theory, so I won't do out the calculations to all orders. But I want to at least suggest that like gauge invariance for massless spin-1 fields, general covariance will help resolve the tension between unitarity and Lorentz invariance.

4 The Graviton Propagator from Cutting Rules

As we saw in the gauge theory example, the propagator for our boson is determined by the sum over polarizations of the physical external states. So first we need to know what the physical polarizations are for a spin-2 field.

Sourceless, classical gravity is described by the Einstein equation,

$$G_{ab} = 0. \tag{30}$$

If we expand $g_{\mu\nu} = \eta_{\mu\nu} + 2\lambda h_{\mu\nu}$ in plane waves

$$h_{\mu\nu} = \epsilon_{\mu\nu}(p)e^{ipx}, \tag{31}$$

then in momentum space, it turns out that the Einstein equation is equivalent to the following conditions on $\epsilon_{\mu\nu}(p)$:

$$p^\mu \epsilon_{\mu\nu} = 0, \qquad \epsilon^\mu_\mu = 0. \tag{32}$$

(Recall that $\epsilon_{\mu\nu}$ has to be symmetric in μ and ν because $h_{\mu\nu}$ is.) Just like in the photon case, Einstein's equation is invariant under shifts:

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \alpha_\mu p_\nu + \alpha_\nu p_\mu + \cdots \tag{33}$$

as long as $\alpha_\mu p^\mu = 0$. The dots indicate higher order terms in α. It's not hard to check that to first order, this symmetry is the momentum space equivalent of general covariance

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}. \tag{34}$$

Just as in the case of a massless spin-1 field, a Lorentz boost on $\epsilon_{\mu\nu}$ will in general give terms proportional to $\alpha_\mu p_\nu + \alpha_\nu p_\mu$. Therefore, if we want physical amplitudes to be Lorentz invariant while still being independent of the three non-existent longitudinal modes, we must have that

$$p_\mu \mathcal{M}^{\mu\nu\cdots}(p, \cdots) = p_\nu \mathcal{M}^{\mu\nu\cdots}(p, \cdots) = 0 \tag{35}$$

for each external graviton and any number of other external particles. (By the symmetry of $\mathcal{M}^{\mu\nu}$ we really only need one of the these conditions.)

Counting the polarization degrees of freedom:

$$\begin{array}{rrl} & 10 & \text{degrees of freedom for symmetric } \epsilon_{\mu\nu} \\ - & 5 & \text{constraints from } p^\mu \epsilon_{\mu\nu} = 0 \text{ and } \epsilon^\mu_\mu = 0 \\ - & 3 & \text{unphysical degrees of freedom from } \alpha_\mu \text{ degeneracy} \\ \hline & 2 & \text{physical polarizations for a massless spin-2 field} \end{array} \tag{36}$$

For example, for a graviton with momentum

$$k^\mu = (E, 0, 0, E), \tag{37}$$

the two polarization states can be

$$\epsilon^1_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon^2_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

Our normalization condition is $\epsilon^a_{\mu\nu} \epsilon^{b\mu\nu} = \delta^{ab}$. We want to figure out what the sum-over-polarizations yields. If we let

$$\tilde{k}^\mu = (E, 0, 0, -E), \tag{39}$$

then with a bit of algebra we can show that

$$\sum_a \epsilon^a_{\mu\nu} \epsilon^a_{\rho\sigma} = \frac{1}{2} \lambda_{\mu\rho} \lambda_{\nu\sigma} + \frac{1}{2} \lambda_{\mu\sigma} \lambda_{\nu\rho} - \frac{1}{2} \lambda_{\mu\nu} \lambda_{\rho\sigma}, \tag{40}$$

where

$$\lambda_{ab} = \eta_{ab} - \frac{k_a \tilde{k}_b + \tilde{k}_a k_b}{k \cdot \tilde{k}}. \tag{41}$$

We saw already that $p_\mu \mathcal{M}^{\mu\nu} = 0$. So in analogy with the massless spin-1 case, we can drop all dependence on $\tilde{k}_\mu$ and just consider Lorentz invariant combinations of $\eta_{\mu\nu}$ and k_μ. A general graviton propagator consistent with the cutting rules is

$$\mu\nu \;\sim\!\sim\!\sim\!\sim\; \rho\sigma = \frac{i}{k^2 + i\epsilon}\left(\frac{1}{2}\lambda^{(1)}_{\mu\rho}\lambda^{(1)}_{\nu\sigma} + \frac{1}{2}\lambda^{(1)}_{\mu\sigma}\lambda^{(1)}_{\nu\rho} - \frac{1}{2}\lambda^{(2)}_{\mu\nu}\lambda^{(2)}_{\rho\sigma} + \alpha^3 \frac{k_a k_b k_c k_d}{k^4}\right), \tag{42}$$

where now

$$\lambda^{(i)}_{ab} = \eta_{ab} - \alpha^i \frac{k_a k_b}{k^2} \tag{43}$$

and α^1, α^2, and α^3 are arbitrary constants that play the same "gauge fixing" role as ξ in the photon case. Indeed, there is one arbitrary constant for each unphysical polarization state. (The fact that the $\alpha' s$ are grouped the way they are comes from the fact that the propagator should be invariant under $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$, and $\mu\nu \leftrightarrow \rho\sigma$.) For doing calculations, we'll usually go to the equivalent of Feynman gauge where $\alpha^1 = \alpha^2 = \alpha^3 = 0$.

$$\mu\nu \;\sim\!\sim\!\sim\!\sim\; \rho\sigma = \frac{i}{k^2 + i\epsilon}\left(\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{1}{2}\eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right) \tag{44}$$

If we were really enthusiastic, we could figure out the propagator for a massive graviton by summing over all five polarizations. See appendix A for that calculation, where we find a bit of a surprise.

5 The First-Order Structure of the Lagrangian

Now that we've figured out the propagator for gravity, we could invert it to find the corresponding quadratic terms in the Lagrangian. We will find that the terms correspond to the $\mathcal{O}(\lambda^0)$ terms in the expansion of

$$\mathcal{L}_{gravity} = \frac{1}{\lambda^2}\int d^4x\, \sqrt{g}R, \tag{45}$$

plus three gauge fixing terms corresponding to the α_i's. But what about the $\mathcal{O}(\lambda^1)$ and $\mathcal{O}(\lambda^2)$ terms? Where do they come from?

In the absence of other fields, there is no good way to proceed. That's because gravity is self-consistent just with the quadratic terms, though it is a boring theory with no interactions. It is certainly possible to declare that there are three- and four-graviton vertices and try to figure out how to make a self consistent theory out of them. But it is more instructive to look at couplings to matter.

In classical electromagnetism, we can write the Lagrangian as:

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu, \tag{46}$$

where j^μ is the electromagnetic current. We want this current to be conserved, so

$$\partial_\mu j^\mu = 0. \tag{47}$$

For fermion fields, $j^\mu = \sum_i q_i \bar{\psi}_i \sigma^\mu \psi_i$ where q_i is the charge of the particle. Note that j^μ is just the Noether current of the fermion field $U(1)$ symmetry and is independent of A_μ.

For a complex scalar field, however, we might naively guess that $j^\mu = q\phi^*\partial_\mu\phi + h.c.$, because this is the current associated with the $U(1)$ symmetry that we will gauge. However, we know from unitarity arguments (or from gauge invariance) that there is a $q^2\phi^*\phi A_\mu A^\mu$ coupling in Lagrangian, so for self-consistency, we find that j^μ must also be a function of A_μ.

Similarly, we might guess that the graviton should couple to matter via

$$\mathcal{L}_{int} = \lambda h_{\mu\nu} T^{\mu\nu}, \tag{48}$$

where $T^{\mu\nu}$ is some symmetric two-index conserved current associated only with the matter fields:

$$\partial_\mu T^{\mu\nu} = 0. \tag{49}$$

One such symmetric two-index conserved current is the energy momentum tensor, which is the Noether current of space-time translation symmetry. For a real scalar field

$$\mathcal{L}_{scalar} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

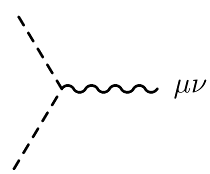
$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right). \tag{50}$$

So our hypothesized matter-gravity interaction is

$$\mathcal{L}_{int} = \lambda h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}h\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}hm^2\phi^2, \tag{51}$$

where $h = h^\mu_\mu$. Note that this is the same interaction Lagrangian from the generally covariant expansion in equation (3). To first-order, this will be the correct interaction, but from the lesson of the complex scalar field in QED, we expect that at higher order, $T^{\mu\nu}$ would be a function of $h_{\mu\nu}$. This is equivalent to saying that in order for energy-momentum to be conserved, we need to take into account the energy contribution from the graviton field itself. We'll see this momentarily in the language of unitarity.

We can summarize our first-order interaction in the vertex:



(52)

You might ask how we know that there are no interactions with two gravitons and two matter fields. The answer is we don't. We might try to have an interaction like $h^2\phi^2$, and we would have to check whether such an interaction would violate unitarity. Similarly, we might wonder gravity should couple to the energy-momentum tensor, when there are a lot of other two-index objects (not necessarily classically conserved) that we can form. Again, unitarity will tell us what the allowed interactions are, and we would find that unitarity requires the terms in our Lagrangian to be general covariant. That is, the Lagrangian for matter will end up being

$$\mathcal{L}_{matter} = \sqrt{g}\, f(g_{\mu\nu}, \phi, \cdots). \tag{53}$$

If we define

$$T_{\mu\nu} \equiv \sqrt{g}\,\frac{\delta\mathcal{L}_{matter}}{\delta g_{\mu\nu}}, \tag{54}$$

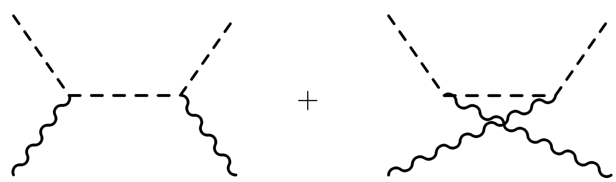
then the interaction Lagrangian is roughly equivalent to (in some linearization)

$$\mathcal{L}_{int} \simeq \lambda h_{\mu\nu} T^{\mu\nu} + \cdots . \tag{55}$$

A good way to motivate coupling gravity to the energy-momentum tensor is that such a coupling reproduces Newtonian gravity in the non-relativistic limit. That is, we want gravity to couple to mass, and we know that mass is conserved in this limit. Of course, if we want to create a generic spin-2 theory, we might ask why we didn't try writing down every Lorentz invariant term and seeing what relationship between the couplings were necessary. We would end up at the same result, but we are able to short-circuit this tedious calculation by, in a sense, already knowing the answer from Einstein.

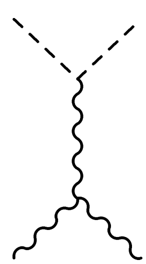
At this point, I should also mention that unitarity requires universal coupling. That is, every matter field must have the same interaction strength (*i.e.* coupling constant) with gravity. The proof using soft gravitons in found in Weinberg [3]. This means that if one field gravitates, then all fields (including gauge bosons) must gravitate.

Assuming we agree that there should be (or at least could be) a gravtion-scalar-scalar vertex, we can see that there must also be a three-graviton vertex. The Compton scattering of a scalar looks like



(56)

If we calculate $k_\mu \mathcal{M}^{\mu\nu\cdots}(k,\cdots)$ for this process, we find that it isn't zero. If we add a graviton-graviton-scalar-scalar interaction, this doesn't give us the desired cancellation either. The only other option is a three-graviton vertex, which would allow diagrams like



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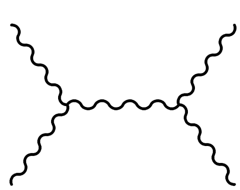
It would be incredibly tedious to figure out all of the three graviton interactions using this method because there are something like eighteen different Lorentz invariant terms we could add to our Lagrangian at this order. We would probably need to look at more than just Compton scattering to figure out each coefficient, and I'm not even sure if anyone has done this calculation explicitly.

Then, once we had our three-graviton vertex nailed down, we would have to consider four graviton interactions.



$+$ *t-channel* $+$ *u-channel* (58)

Calculating $k_\mu \mathcal{M}^{\mu\nu\cdots}(k,\cdots)$ would again give a non-zero result, and we would have to introduce a four-graviton vertex to cancel this unitarity violation. This would continue to all orders, but what we would find is that our Lagrangian would just yield the λ expansion of

$$\mathcal{L}_{gravity} = \frac{1}{\lambda^2}\int d^4x\,\sqrt{g}R. \tag{59}$$

(Well, this isn't exactly true. Once we hit $\mathcal{O}(\lambda^3)$, we have the freedom to add terms like $R_{\mu\nu}R^{\mu\nu}$ to our Lagrangian.) Of course, this is all very tedious algebra, and Feynman shows a shortcut in his Gravitation lecture notes [1]. In the end, we find what we knew all along: a unitary Lagrangian for gravity will have a form that "happens" to exhibit general covariance symmetry. Presumably, if you worked really hard, you could prove a kind of Ward identity for gravity that would show that any theory with general covariance symmetry would have vanishing $k_\mu \mathcal{M}^{\mu\nu\cdots}(k,\cdots)$.

6 Looking Ahead

You may be a bit disappointed that the principle of general covariance as a geometric construct is obscured in the language of effective field theory. In Feynman's lecture notes he goes through all of the algebra to show what the Lagrangian has to look like, and he shows that you have to introduce the Ricci scalar in the Lagrangian to get a consistent theory of gravitational interactions. But the form of the Ricci scalar is determined not from geometric reasoning but from cancellation of pieces that give a non-zero contribution to $k_\mu \mathcal{M}^{\mu\nu\cdots}(k,\cdots)$. The fact that R has geometric significance is completely lost.

We could say the same about gauge invariant theories. Working in the effective field theory language, gauge invariance has nothing to do with turning a global symmetry into a local symmetry. Rather, gauge invariance (with gauge fixing terms) helps guarantee that our theory will be unitary. In a sense, we are trading the language of symmetries for the language of consistent theories.

In the end, we are ultimately interested in *finding* consistent theories, so it is useful to use general covariance and gauge invariance to help write down such theories without having to calculate a multitude of Compton scattering amplitudes. Similarly, by using the effective field theory language, it is easy to see that we can do quantum gravity calculations despite the non-renormalizability of gravity. As long as our energy scales are less than $1/\sqrt{\lambda}$ we can do perturbative calculations until we are blue in the face. Gravity is merely a theory a spin-2 field whose interactions and self-interactions are determined by the requirement of unitarity, and whose coupling constant happens to have negative mass dimension.

From here, we might want to know about massive gravitons and the specific structure of the mass terms in the Lagrangian. We might want to know whether it is possible to have two or more gravitons inhabiting the same space. And we might like to know if we could deconstruct gravity in the same way we can deconstruct gauge theories. These are the issues I'd like to turn to for the remainder of the summer. My hope is that by understanding a theory as strongly coupled as gravity, I can get a better feel for the kind of gymnastics one needs to form successful models in high energy physics.

A Propagator for Massive Graviton

We can use the same sum-over-polarizations rule from section 4 to find the propagator for a massive graviton. Just like equation (32), the polarizations for a massive graviton must satisfy $p^\mu \epsilon_{\mu\nu} = 0$ and $\epsilon^\mu_\mu = 0$. Now, however, the longitudinal modes of graviton are physical states, so we have a total of five physical polarizations for a massive graviton.

By Lorentz invariance, we can work in the rest frame of the graviton without loss of generality. Let

$$k_\mu = (m, 0, 0, 0). \tag{60}$$

A polarization basis that satisfies the normalization $\epsilon^a_{\mu\nu}\epsilon^{b\mu\nu} = \delta^{ab}$ is

$$\epsilon^1_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon^2_{\mu\nu} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$\epsilon^3_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon^4_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \epsilon^5_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{61}$$

Now we can show that

$$\sum_a \epsilon^a_{\mu\nu}\epsilon^a_{\rho\sigma} = \frac{1}{2}\lambda_{\mu\rho}\lambda_{\nu\sigma} + \frac{1}{2}\lambda_{\mu\sigma}\lambda_{\nu\rho} - \frac{1}{3}\lambda_{\mu\nu}\lambda_{\rho\sigma}, \tag{62}$$

where

$$\lambda_{ab} = \eta_{ab} - \frac{k_a k_b}{m^2}. \tag{63}$$

So the propagator for a massive graviton is

$$\mu\nu \;\sim\!\sim\!\sim\!\sim\; \rho\sigma = \frac{i}{k^2 - m^2}\left(\frac{1}{2}\lambda_{\mu\rho}\lambda_{\nu\sigma} + \frac{1}{2}\lambda_{\mu\sigma}\lambda_{\nu\rho} - \frac{1}{3}\lambda_{\mu\nu}\lambda_{\rho\sigma}\right) \tag{64}$$

Note the factor of $\frac{1}{3}$ instead of $\frac{1}{2}$ in the last term of the propagator. This is related to the vDVZ discontinuity [5], which tells us that experimental predictions from massless gravity differ from the $m \to 0$ limit of massive gravity.

References

[1] R. Feynman. *Lectures on Gravitation.* Caltech, 1971. (QC178 F49)

In these notes, Feynman develops gravity as a generic theory of a massless spin-2 field. There is an amusing extended introduction in which he gives as overview of a bunch of wacko (and falsified) ideas about how gravity could work. Once he constructs the Einstein Lagrangian, his treatment of gravity follows that of most GR textbooks. There is also a revised version (*c.* 2003) of these notes available.

[2] N. Arkani-Hamed. QFT Lecture Notes, Spring 2003.

This was my inspiration for analyzing the tension between Lorentz invariance and unitarity for gravity. In Nima's class, we saw that a manifestly Lorentz invariant theory of a massless spin-1 particle had to have a gauge symmetry (and a gauge-fixing term) in order to be unitary. I wanted to see how this worked in gravity to understand why general covariance was a necessary symmetry.

[3] Weinberg, Vol. 1, Chapter 13.

In this chapter, Weinberg proves universal coupling using soft photons.

[4] R. Feynman. "Quantum Theory of Gravitation." *Acta Physica Polonica*. **24**, 697 (1963).

I include this reference more for historical reasons than for its content. In my search for papers on gravity and unitarity I stumbled across this transcript from a talk at the *Conference on Relativistic Theories of Gravitation*. It offers a glimpse at how Feynman conceived of quantum gravity in the early stages of research. It will also give you a chance to enter the dusty but strangely charming Journal Archive Room.

[5] H. van Dam & M. Veltman. "Massive and Mass-less Yang-Mills and Gravitational Fields." *Nuclear Physics B*. **22**, 397 (1970).

I had calculated the propagator for the massive graviton before looking at this, the original paper (along with the concurrent paper by Zakharov) on the vDVZ discontinuity. It was nice to know that Mathematica had indeed given me the correct answer. This paper also has an appendix on linearized gravity.