Mooses, Anomalies, and Deconstruction

Jesse Thaler

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1 Motivation

This summer we want to understand extra dimensions, and one interesting model of extra dimensions is deconstruction. In deconstruction, one or more extra dimensions appear at *low* energy, whereas at high energy the model reveals itself as an asymptotically free gauge theory in four dimensions. When we think of compact dimensions or brane-world scenarios, we often think of extra dimensions as appearing only when we go to high energy. In deconstruction, the extra dimension appear as part of the confinement of a series of linked $SU(N)$ gauge groups. Thus, to understand deconstruction, we need to understand how to create these linked QCD-like theories.

The tool we will use is the moose diagram. Moose diagrams have been used in various contexts, but Georgi [1] was motivated to use them by a search for models that yield massless composite fermions. For example, we know that the proton is a composite fermion formed out of three quarks, but the proton is certainly not massless, with its mass given roughly by the confinement scale. Georgi wanted to find a confinement mechanism that yielded composite fermions whose quantum numbers *forbid* a mass term in the low energy effective theory.

But just drawing a random moose diagram is not enough. We need to make sure that our moose is anomaly-free in order to claim that our moose is a valid quantum theory. We will see that this will necessitate choosing the matter content of our theory carefully to guarantee that all anomalies cancel. Because most of us aren't all that comfortable with anomalies yet, I will spend a significant amount of time showing a quick and dirty way to understand where a chiral anomaly could come from. (Contrast this with the approach in [2].)

Then, once we are certain that we have an anomaly-free moose, we will be able to see quite clearly how confinement works in these models. In the particular case of deconstruction [3], above the confinement scale we will have a standard gauge theory of Weyl fermions that's asymptotically free (because it's a non-Abelian gauge theory with sufficiently low matter content) and anomaly-free (because we chose it to be that way). Below the confinement scale, we will have a theory of gauge sites and link fields that for most intents and purposes looks like a five dimensional gauge theory with a latticized compact fifth dimension. In this way, we will see the link between moose model building and extra dimensions.

2 Moose Diagrams

A moose diagram is simply an array of sites with links. The sites represent gauge or global symmetry groups, and the links represent Weyl fermion fields. As we know, once we specify the gauge groups and the matter content, we can easily write down the Lagrangian that corresponds to that model. For example, the moose

$$\underline{SU(M)} \xrightarrow{\psi_1} SU(N) \xrightarrow{\psi_2} \underline{SU(M)} \tag{1}$$

represents M fields ψ_1 with a global $SU(M)$ symmetry transforming under the fundamental representation of $SU(N)$, and M fields ψ_2 with a separate global $SU(M)$ symmetry transforming under the anti-fundamental representation of $SU(N)$. This yields the Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_{i=1}^{M} \overline{\psi}_{1i} i\sigma_\mu D_\mu^N \psi_{1i} + \sum_{j=1}^{M} \overline{\psi}_{2j} i\sigma_\mu D_\mu^{\overline{N}} \psi_{2j} \tag{2}$$

where D_μ^N is the covariant derivative for a field transforming in the fundamental representation and $D_\mu^{\overline{N}}$ is the covariant derivative for a field transforming in the anti-fundamental representation.

As we can see, a moose site can either be underlined or not underlined:

$$\begin{aligned} SU(N) &\equiv \text{gauge symmetry group} \\ \underline{SU(N)} &\equiv \text{global symmetry group} \end{aligned} \tag{3}$$

An arrow into a site means that the fermion transforms in the fundamental representation of the relevant site and an arrow out of a site means that the fermion transforms in the anti-fundamental representation.

$$\begin{aligned} SU(N) \longleftarrow\!\!- \quad &\equiv \text{fermion in the } \mathbf{N} \text{ representation} \\ SU(N) \longrightarrow\!\!- \quad &\equiv \text{fermion in the } \overline{\mathbf{N}} \text{ representation} \end{aligned} \tag{4}$$

Note that in this moose notation, there is no easy way to designate a fermion field to transform under three or more symmetries because an arrow only has two ends. Note also that our fermion fields can only transform in the fundamental or anti-fundamental representations of a group. Well, you could also have a fermion transform in the adjoint representation by having a moose with an arrow going into and out of the same site:

(5)

This works because we know that for $SU(N)$

$$\textit{fundamental} \otimes \textit{anti-fundamental} = \textit{adjoint} \oplus \textit{singlet}. \tag{6}$$

But you won't get anywhere with the moose if you want to use obscure higher dimensional representations. (Or even something as simple as the **16** of $SO(10)$.

As a goofy illustration of the moose, I will draw the moose for one generation of the standard model. Note that there is no simple way to write down hypercharges with the moose, either.

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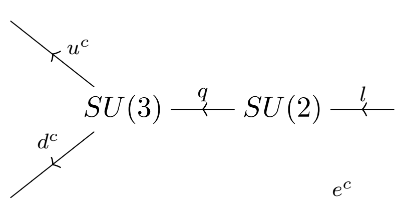


What is interesting about this moose is that it makes no reference to the Higgs mechanism. Even if we put a dashed line into the $SU(2)$ site to indicate the Higgs doublet, we wouldn't necessarily know that the Higgs gets a vev which spontaneously breaks $SU(2)$ in such a way that the low energy effective theory consists of Dirac spinors u, d, and e and Weyl spinor ν. The point is that moose diagrams are used when symmetries are broken through QCD-like confinement, not spontaneous symmetry breaking à la Higgs.

Before talking about confinement in the moose, though, how do we know that we can really trust the predictions of our moose theory? For an $SU(N)$ theory with large enough N and small enough matter content, we know that the theory will be asymptotically free, so it makes sense deep into the UV. If we trust the predictions of QCD, then by asymptotic freedom we should trust the predictions of our moose. But there's one other thing we should worry about: anomalies.

3 Detour to Anomalies

In order to understand how to generate anomaly-free mooses, we will need to understand what anomalies are an how to cancel them. If you zone out once we begin the anomaly discussion in the next section, then all you need to know for the remainder of this talk is the following: a moose with non-Abelian gauge interactions is anomaly-free if the same number of arrows go into and out of each gauge site. For example, an $SU(N)$ gauge theory is anomaly-free if there are k fermions transforming in the $\mathbf{N}$ representation and k fermions transforming in the $\overline{\mathbf{N}}$ representation.

There are subtleties when we have an Abelian $U(1)$ symmetry. (This is the reason why you often hear that the Standard Model has the "unique" anomaly-free matter content allowed by a $SU(3) \times SU(2) \times U(1)$ gauge theory. The subtleties with the $U(1)$ gauge anomalies fix the allowed hypercharges for the chiral content.) We'll understand these subtleties in a moment, but let's first understand what restriction anomalies place on moose diagrams.

Consider the following moose:

$$\cdots \longrightarrow SU(L) \xrightarrow{\psi_1} SU(M) \xrightarrow{\psi_2} SU(N) \longrightarrow \cdots \tag{8}$$

Here ψ_1 transforms in the $(\overline{\mathbf{L}}, \mathbf{M}, \mathbf{1})$ representation and ψ_2 transforms in the $(\mathbf{1}, \overline{\mathbf{M}}, \mathbf{N})$ representation. So there are L fermions in the fundamental representation of $SU(M)$ and N fermions in the anti-fundamental representation of $SU(M)$. For this moose to be anomaly-free, we must have $L = N$. A linear moose diagram (one with no branching) will *have* to be a chain of gauge groups that alternate $SU(N)$, $SU(M)$, $SU(N)$, $SU(M)$, and so on. (Insert your favorite gauge group if you are sick of special unitary matrices.)

Similarly, consider this non-linear moose:

$$\begin{array}{ccccc} \cdots \longrightarrow SU(L) & \xrightarrow{\psi_1} & SU(M) & \xrightarrow{\psi_2} & SU(N) \longrightarrow \cdots \\ & & \psi_3 \big\uparrow & & \\ & \cdots \longrightarrow & SU(K) & & \end{array} \tag{9}$$

This has $L + K$ fermions transforming in the fundamental representation of $SU(M)$ and N fermions transforming in the anti-fundamental representation of $SU(M)$, so for this moose to be anomaly-free, we must have $L + K = N$. Of course, you could imagine having fermions that transform in more exotic representations of $SU(N)$ (or even more exotic gauge groups). In those cases, you would have to be more careful about anomaly cancellations and will most likely have to calculate triangle diagrams to check that everything works out. As mentioned already, every moose that you will encounter will only have fermions that transform in the fundamental or anti-fundamental representations of the relevant gauge groups.

There are certain cases where the gauge group we are considering has a real (as opposed to complex) fundamental representation. Consider, for example the **N** of any $SO(N)$ or the **2** of $SU(2)$. In those cases, the arrows are meaningless, and in particular, there are no anomalies associated with lines with no arrows. So the following moose is fine:

$$SO(N) \longrightarrow\!\!\!- \underline{SU(M)} \tag{10}$$

Recall that underlining a group means that the group is only a global symmetry not a gauge symmetry. The arrow on the line refers only to the global $SU(M)$ and is meaningless for the gauged $SO(N)$.

But we still haven't answered the question: what are anomalies and why are they associated with triangle diagrams? An anomalous symmetry is a symmetry of the Lagrangian that holds when we treat the fields as classical fields, but breaks down when we treat the fields as quantum fields. As Nima remarked in his last QFT lecture, these anomalies appear in the IR as non-trivial boundary gauge field configurations. In the UV, anomalies make it impossible to make a guess for the UV physics (*i.e.* choose a regulator for the path integral) that respects the classical symmetry.

There are two types of anomalies: anomalies that violate global symmetries and anomalies that violate gauge symmetries. Global anomalies are certainly important for understanding the quantum structure of a theory, but no one is going to be too upset if the chiral symmetry that protects, say, the mass of a Weyl fermion turns out to be anomalous. Gauge anomalies, however, are absolutely disastrous. The unitarity of a gauge theory rests on the existence of the gauge symmetry, so if that symmetry is fundamentally incompatible with a quantum regulator, then the gauge theory won't make sense as a quantum theory. Therefore, we want our gauge theories to be anomaly-free, and in order to do that, we have to choose the chiral content of our theories very carefully.

4 Chiral Anomalies

Consider a general gauge theory of chiral fields. The matter part of the Lagrangian looks like:

$$\mathcal{L}_{matter} = \sum_i \overline{\psi}_i i\sigma^\mu D_\mu \psi_i. \tag{11}$$

Because this is a gauge theory, we have the standard gauge symmetry. In particular, we can look at the global version of the gauge symmetry under which

$$\psi_i \to e^{i\theta^a T_a^{(i)}} \psi_i, \qquad \delta_a \psi_i = i T_a^{(i)} \psi_i \tag{12}$$

where $T_a^{(i)}$ is the generator of the gauge theory corresponding to the representation of the i-th field. Using Noether's theorem, we can derive the conserved current associated with this global symmetry.

$$j_a^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \delta_a \psi_i = -\sum_i \overline{\psi}_i \sigma^\mu T_a^{(i)} \psi_i \tag{13}$$

It's often convenient to drop the minus sign in the definition of j_a^μ.

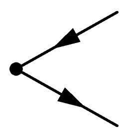
Classically, we know that $\partial_\mu j_a^\mu = 0$, but what if this global symmetry is anomalous? Because this global rotation is a subset of the full gauge symmetry, if $\partial_\mu j_a^\mu \neq 0$ at the quantum level, then this anomaly violates the gauge symmetry that keeps the theory unitary. So it is imperative that we check to make sure that j_a^μ is conserved (at least) perturbatively.

Treating j_a^μ as an operator in momentum space, we want check if

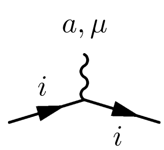
$$iq_\mu \cdot \langle ext|\, j_a^\mu(q_\mu)\, |0\rangle \overset{?}{=} 0 \tag{14}$$

for all external states. This is equivalent to showing that the Noether current is conserved quantum mechanically. Let's check this to one loop order.

In perturbation theory, the operator j_a^μ looks like the vertex

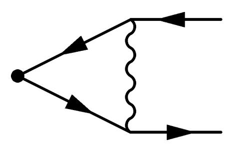
$j_a^\mu \rightarrow$  $= \sum_i \sigma^\mu T_a^{(i)}.$ (15)

Classically, we know that the divergence of the operator vanishes, so we expect that at tree level, the divergence of this vertex when applied with two external fermion lines should vanish. The anomaly is a quantum effect, so we need to form loops somehow. The only other vertex relevant to our calculation is the standard gauge-fermion vertex:



$= ig\sigma^\mu T_a^{(i)}$ (16)

We can try forming the following loop with j_a^μ:



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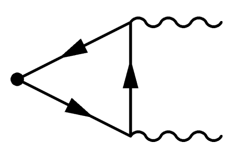
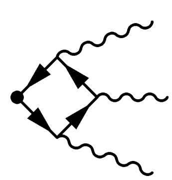
It turns out that with fermion legs, there is no anomaly. What about with external gauge bosons?

 $+$ *crossed* (18)

We will see that a possible source for an anomaly is from triangle diagrams like these with two external gauge boson legs. Of course, you could imagine a diagram like:



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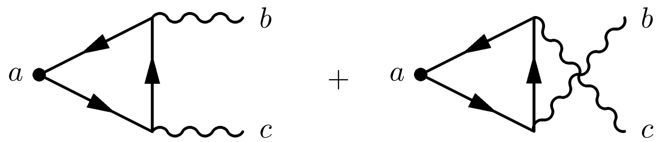
This would give additional contributions to the anomaly, but for our purposes, it is sufficient to focus on the triangle anomaly.

Now, at this point, we could go and calculate the possible anomaly contribution using a careful application of dimensional regularization. The argument is presented in Peskin and Schroeder, Chapter 19, but they insist on using Dirac fermions, which I think masks the real argument. As we'll see momentarily, any theory of Dirac fermions is necessarily free of chiral gauge anomalies, so they have to do a lot of γ_5 nonsense to isolate the left and right chiral components.

With Weyl fermions, we can see that in general we don't expect a cancellation of the two diagrams below:



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In fact, we expect that they should roughly be proportional to:

$$\langle ext|\, \partial_\mu j^\mu_a(x_\mu)\, |0\rangle \propto \sum_i \mathrm{tr}(T_a^{(i)} T_b^{(i)} T_c^{(i)} + T_a^{(i)} T_c^{(i)} T_b^{(i)}). \tag{21}$$

(I guess it's less than obvious that there should be a plus sign between these two terms. All I can say is that crossing the diagram doesn't introduce any new factors.) The trace comes from the fermion loop and the group generators come from the Feynman rules for j^μ_a and the gauge-fermion vertex. We might hope that dotting iq_μ into the amplitude might help things (à la the Ward identity), but it turns out that this doesn't work. The quantity

$$\mathcal{A}^{(i)}_{abc} = \mathrm{tr}(T_a^{(i)}\{T_b^{(i)}, T_c^{(i)}\}) \tag{22}$$

determines the anomaly contribution from the i-th fermion, and the relation

$$\sum_i \mathcal{A}^{(i)}_{abc} \stackrel{?}{=} 0 \tag{23}$$

determines whether the theory is anomalous or not. Now we see why the choice of the matter content of a theory can drastically affect unitarity.

Let's say that we have a $SU(N)$ gauge theory of $2k$ chiral fields, k in representation $\mathbf{N}$ and the other k in representation $\overline{\mathbf{N}}$. Can we write $\mathcal{A}^{\overline{N}}_{abc}$ in terms of $\mathcal{A}^{N}_{abc}$? If T^N generates the transformation

$$\psi \to (1 + i\theta^a T^N_a)\psi, \tag{24}$$

then taking the complex conjugate to find the generators of the conjugate representation

$$\psi^* \to (1 - i\theta^a (T^N_a)^*)\psi^* \equiv (1 + i\theta^a T^{\overline{N}}_a)\psi^*. \tag{25}$$

The generators of a Lie Algebra are Hermitian, so

$$T^{\overline{N}}_a = -(T^N_a)^* = -(T^N_a)^{\dagger *} = (-T^N_a)^T. \tag{26}$$

Calculating $\mathcal{A}^{\overline{N}}_{abc}$:

$$\mathcal{A}^{\overline{N}}_{abc} = \mathrm{tr}(T^{\overline{N}}_a\{T^{\overline{N}}_b, T^{\overline{N}}_c\}) = \mathrm{tr}\left((-T^N_a)^T\{(-T^N_b)^T, (-T^N_c)^T\}\right) = -\mathrm{tr}(\{T^N_c, T^N_b\}T^N_a) = -\mathcal{A}^N_{abc} \tag{27}$$

We see right way that we have the cancellation

$$k \cdot \mathcal{A}^N_{abc} + k \cdot \mathcal{A}^{\overline{N}}_{abc} = 0, \tag{28}$$

so as advertised, a gauge theory with equal number of chiral fields in the fundamental and anti-fundamental is anomaly-free. Similarly, recall that a Dirac fermion can be written as

$$\psi = \begin{pmatrix} \psi_L \\ \overline{\psi}_R \end{pmatrix}, \tag{29}$$

so if ψ transforms in representation $\mathbf{R}$, ψ_L transforms in $\mathbf{R}$ and ψ_R transforms in $\overline{\mathbf{R}}$. Therefore, the theory is anomaly-free if all the fermions are grouped into Dirac spinors. Continuing, we can use the same algebra above to convince ourselves that a real representation will never contribute to the anomaly because $\mathcal{A}^{Re}_{abc} = -\mathcal{A}^{Re}_{abc} = 0$.

Here are some facts about Lie Algebras that you might find interesting. The only simple groups with complex representations (and therefore the only simple groups that can have anomalies) are $SU(n)$, $S0(4n+2)$, and E_6. It turns out that the quantity $\mathcal{A}_{abc}$ is proportional to a symmetric group invariant d_{abc}, and from the above groups, the only ones that have such an invariant (and therefore could be anomalous) are $SU(n)$ and $SO(6)$. In particular, gauge theories based on $SO(4n + 10)$ for $n \geq 0$ and E_6 can have *arbitrary* matter content and remain anomaly-free, even if there are unmatched complex representations. Therefore, you might try cooking up a unified theory out of, say, $SO(10)$.

Finally, a word about $U(1)$ gauge theories. Clearly, the cube of the $U(1)_Y$ charges have to sum to zero because for a $U(1)$ symmetry we have $\mathcal{A}^{(i)}_{YYY} = q_i^3$. If we have a $U(1)_Y \times SU(N)$ gauge theory, then we also have terms like $\mathcal{A}(i)_{Yaa} = 2q_i \mathrm{tr}(T^{(i)}_a T^{(i)}_a) = 2q_i C(i)$ whose sum over i has to vanish. (Note that $\mathcal{A}(i)_{YYa}$ is zero for any simple group because the trace of each generator vanishes.) The subtlety comes in when instead of having gauge bosons on the external legs of our triangle diagram, we have gravitons on the external legs. In that case, we have something like $\mathcal{A}^{(i)}_{agg} \propto \mathrm{tr}(T^{(i)}_a)$. For simple non-Abelian groups this vanishes, but for $U(1)$, we have to satisfy the additional constraint that the sum of the $U(1)$ charges must be zero.

5 Confinement and the Moose

Now we are ready to talk about confinement in moose diagrams. We are all familiar with QCD-like theories like the following moose:

$$\underline{SU(M)_L} \xrightarrow{\psi_1} SU(N)_C \xrightarrow{\psi_2} \underline{SU(M)_R} \tag{30}$$

As we descend in energy to the confinement scale Λ_{QCD}, the only degrees of freedom which can remain are $SU(N)$ singlets. In particular, we will have a spectrum of "hadrons" with masses on the order of Λ_{QCD}, and we will form the condensate $\langle\psi_1\psi_2\rangle$. This condensate will break the global $SU(M)_L \times SU(M)_R$ to the diagonal subgroup $SU(M)_D$. The goldstones from this spontaneous symmetry breaking will yield an $SU(M)$'s worth of pions. Note that all of the fermions in this model are massive, whereas the massless goldstones are bosons.

Georgi was motivated to use the moose to try to find a QCD-like theory that yielded *massless* fermions below the confinement scale. We know that the Standard Model has only massless fermions above the Higgs scale, so it is tempting to try to find out whether the Standard Model fermions could be composite fermions in the same way that the baryons are composite fermions. But QCD alone only produces fermions with masses of the confinement scale and the spectrum of massless particles are bosons, not fermions.

Consider the following extension of QCD:

$$\underline{SU(M)_L} \xrightarrow{\psi_1} SU(N)_{strong} \xrightarrow{\psi_2} SU(M)_{weak} \xrightarrow{\psi_3} \underline{SU(N)_R} \tag{31}$$

Note that from our anomaly discussion, the order of the global and gauge groups are completely fixed by the requirement of anomaly cancellation. The labels "strong" and "weak" tell us that the confinement scale of $SU(N)_{strong}$ is much greater than the confinement scale $SU(M)_{weak}$.

At the confinement scale Λ_{strong}, we form a spectrum of "hadrons" and the condensate $\langle\psi_1\psi_2\rangle$. This condensate will break the $SU(M)_L \times SU(M)_{weak}$ down to a diagonal $SU(M)_D$, but because $SU(M)_{weak}$ is gauged, some of Goldstone bosons from the spontaneous symmetry breaking will be eaten by $SU(M)_{weak}$ to form the longitudinal modes of the now massive gauge field. In fact, *all* of the Goldstone bosons will be eaten, because there is an $SU(M)$'s worth of broken symmetry and an $SU(M)$'s worth of longitudinal modes.

It's actually worthwhile to show this explicitly. Under the $SU(M)_{weak}$ gauge transformation, the condensate $\langle\psi_1\psi_2\rangle = 4\pi f^3 U$ transforms like ($f \simeq \Lambda_{strong}/4\pi$):

$$U \to U e^{i\theta_a(x)T^a} \tag{32}$$

We want to form a gauge invariant action involving the unitary U's. (How do we know that it's unitary?) The low energy effective Lagrangian for the U field *has* to be the non-linear sigma model

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}F^2 + f^2\ \mathrm{tr}\left((D_\mu U)^\dagger D^\mu U\right), \tag{33}$$

with $D_\mu U = \partial_\mu U + iUT_aA^a_\mu$. When U takes its vev $\langle U\rangle = \mathbf{1}$, the Lagrangian becomes (in canonical normalization)

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}F^2 + (fg)^2\ \mathrm{tr}(A_\mu A^\mu) \tag{34}$$

which contains the desired gauge boson mass term with $m = fg$.

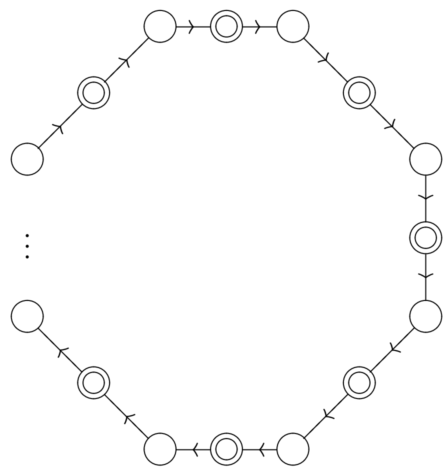
Below Λ_{strong}, we can integrate out the $SU(M)$ hadrons and the $SU(N)_{weak}$ massive gauge bosons. Because all of the goldstones were eaten, all we have left is the fermion field ψ_3. Note that ψ_3 now transforms as a $(\overline{\mathbf{M}}, \mathbf{N})$ under the global $SU(M)_D \times SU(N)_R$ symmetry. Therefore, there is no way to form a mass term $m\psi_3\psi_3$, because it would violate the global chiral symmetry! So we have found a moose that generates massless fermions below the confinement scale.

Of course, if we reverse "strong" and "weak" in the moose, then below the confinement of $SU(M)_{strong}$, we have the fermion field ψ_1 which transforms as a $(\overline{\mathbf{M}}, \mathbf{N})$ under the global $SU(M)_L \times SU(N)_D$ symmetry and therefore is massless. Georgi goes on to show that even if the gauge groups are of comparable strength, the gauge singlet $\langle\psi_1\psi_2\psi_3\rangle$ will become the desired composite fermion with a $(\overline{\mathbf{M}}, \mathbf{N})$ chiral symmetry that protects its mass.

We could go on to explore more mooses with more interesting confinement pictures. All of the linear mooses (*i.e.* ones with no branching) exhibit roughly the same behavior as the models above. To form reasonable models of *interacting* massless composite fermions, we need to go to non-linear mooses, that is, mooses with lots of branching that resemble (duh!) the antlers of a moose. But we aren't done will linear mooses quite yet. What happens if we go to the limit of a large number of gauge sites on a linear moose?

6 Deconstruction

Even better, let's link the gauge sites into a circle with $2\mathcal{N}$ sites ($\mathcal{N}$ $SU(M)$'s and $\mathcal{N}$ $SU(N)$'s). We will see that this moose will become a picture of an extra compact dimension.



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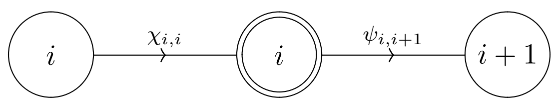
The singly circled sites are weak $SU(M)$'s and the doubly circled sites are strong $SU(N)$'s. For simplicity, we assume that the coupling constants are the same for each type of gauge group. This moose is asymptotically free and anomaly-free for most M and N.

Looking more closely at the i-th side of this moose:



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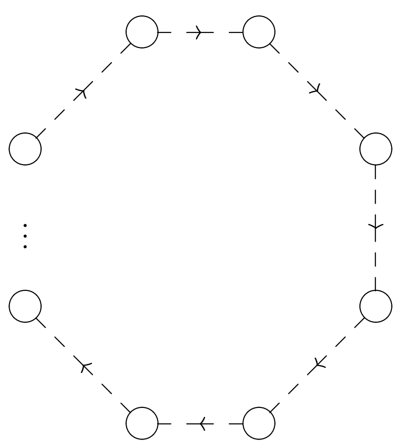
When $SU(N)_i$ gets strong, the χ's and ψ's will have to arrange themselves into $SU(N)_i$ singlets. In analogy with the linear mooses we've already seen, this will produce a spectrum of hadrons with masses around Λ_{strong}. We'll also form the condensate

$$\langle \chi_{i,i}\psi_{i,i+1}\rangle \simeq 4\pi f^3 U_{i,i+1} \tag{37}$$

Again, U is unitary. (Why?) Under the weak $SU(M)$'s, this condensate transforms like

$$U_{i,i+1} = \mathbf{g}_i^{-1}(x)U_{i,i+1}\mathbf{g}_{i+1}(x) \tag{38}$$

where $\mathbf{g}_i$ is an element of the i-th $SU(M)_{weak}$. We can draw a condensed moose to describe the condensed theory (minus the hadrons). The dashed lines will now correspond to a unitary (bosonic) field.



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We want to form a gauge invariant low energy effective Lagrangian for the U's. Our *only* choice is the non-linear sigma model

$$\mathcal{L} = -\frac{1}{2g^2}\sum_{j=1}^{\mathcal{N}} \mathrm{tr} F_j^2 + f^2 \sum_{j=1}^{\mathcal{N}} \mathrm{tr}\left((D_\mu U_{j,j+1})^\dagger D^\mu U_{j,j+1} \right) \tag{40}$$

with $D_\mu U_{j,j+1} = \partial U_{j,j+1} - iA_\mu^j U_{j,j+1} + iU_{j,j+1}A_\mu^{j+1}$. (Recall that $A^j = A_a^j T^a$.)

Before going on, one might ask why we didn't just start with the condensed moose instead of going through this whole argument about condensing a non-Abelian gauge theory. The reason is that the non-linear sigma model breaks down at an energy $4\pi f$, so the condensed moose needs a UV completion. The uncondensed moose is one possible UV completion that not only raises the cutoff above $4\pi f$; it is asymptotically free so it makes sense up to arbitrarily large energies. In a sense, deconstruction will tell us that there exists a UV completion to a gauge theory with a latticized compact fifth dimension, such that predictions made in the low energy theory can be trusted up to operators suppressed by $4\pi f$.

What happens when the U's take their vev? (Equivalently, what happens in unitary gauge?) From what we've seen with other linear mooses, we expect that this will result in a non-trivial mass matrix for the gauge bosons. In particular,

$$\mathcal{L}_{mass} = (gf)^2 \sum_{j=1}^{\mathcal{N}} \mathrm{tr}\left(A^{j\mu}(2A_\mu^j - A_\mu^{j-1} - A_\mu^{j+1}) \right) \tag{41}$$

Following [3], we can diagonalize the mass matrix to arrive at the following spectrum of gauge boson masses:

$$m_k = \left| 2gf \sin\left(\frac{\pi k}{\mathcal{N}}\right) \right| \tag{42}$$

for all integers $|k| \leq \mathcal{N}/2$. In the limit as $\mathcal{N} \to \infty$ while holding $R = \mathcal{N}/gf$ fixed, the masses become

$$m_k \simeq \frac{2\pi|k|}{R} \tag{43}$$

which is a standard Kaluza-Klein spectrum for a compact dimension of circumference R. This is our first hint that our moose is really the picture of an extra dimension.

We can go further and show that our Lagrangian really represents a gauge theory with a compact latticized fifth dimension. The Wilson line from site i to $i+1$ is

$$U_{i,i+1}(x^\mu) = \mathcal{P}\exp\left(i\int_{ai}^{a(i+1)} dx^5 A_5(x^\mu, x^5) \right) \tag{44}$$

where a is the lattice spacing and $\mathcal{P}$ is path-ordering. Under a gauge transformation, the Wilson line transforms as

$$U_{i,i+1}(x_\mu) = \mathbf{g}_i^{-1}(x_\mu) U_{i,i+1}(x_\mu) \mathbf{g}_{i+1}(x_\mu). \tag{45}$$

So to form a gauge invariant Lagrangian out of the Wilson lines, we have to have the same Lagrangian as in eqn. (40). You might wonder why we couldn't have a term in the Lagrangian like

$$\mathcal{L}_? = \mathrm{tr}\left(\prod_{i=1}^{\mathcal{N}} U_{i,i+1} \right) \tag{46}$$

with $\mathcal{N}+1 \equiv 1$. This would represent a Wilson line that went all the way around the extra dimension. It is certainly gauge invariant, but it represents a global rather than local interaction of the gauge field.

Finally, note that by dimensional analysis, the lattice spacing a must be proportional to $1/f$. Recall that the we defined the circumference of the the extra dimension as $R = \mathcal{N}/gf$, so if we say that the circumference is also equal to $\mathcal{N}a$ (*i.e.* the number of sites times the spacing between the sites), we find that $a = 1/fg$. We could go on and show that the four dimensional gauge coupling is related to the five dimensional gauge coupling in the desired way, or that there exists a notion of five dimensional Lorentz invariance. In a sense, all of these follow from the fact that lattice gauge theory reduces to continuum gauge theory in the limit of small lattice spacing. Once we see the Lagrangian in eqn. (40), we know that a fifth dimension has truly emerged.

7 Looking Ahead

Now that's we've developed the language of mooses, we have many directions we could explore. We could return to the idea of QCD-like confinement to try to find a moose diagram that generates the Standard Model at low energies. We could try to use deconstruction to generate more than one extra dimension, or even spaces with non-trivial topology. And, as Rakhi will show at the end of the month, we could extend the notion of gauge sites and link fields to gravity, where the "symmetry" that guarantees unitary is general covariance. She will show that the gravitational moose will not only give us an easy way to understand massive gravitons, but also a way to generate a lattice theory of gravity.

References

[1] H. Georgi. "A tool kit for builders of composite models." *Nucl. Phys.* **B266** (1986) 274.

As far as I can tell, this is the original moose paper. Apparently, moose diagrams were developed independently (?) by S. Dimopoulos, S. Raby, and L. Susskind. (Dimopoulos was the first to dub them "Moose Diagrams.")

[2] Peskin and Schroeder, Chapter 19 and Weinberg, Vol. 2, Chapter 22.

These were my sources for information about anomalies. To my mind, both books take an unnecessarily technical view of anomalies. If anyone knows of a better reference, please let me know.

[3] N. Arkani-Hamed, A. G. Cohen, and H. Georgi. "(De)Constructing Dimensions." hep-th/0104005

The original deconstruction paper.

The Feynman diagrams in these notes were made with feynMF, and the moose diagrams were made with XY-pic. Both of these are standard add-ons to LaTeX, and I'd be happy to show anyone how to use them.