Supersymmetry Basics

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1 Motivation

Is there any reason to expect that our universe is supersymmetric? We could just as well ask if we have any reason to expect our universe to be homogeneous (translationally invariant) or isotropic (rotationally/boost invariant). Supersymmetry is just an extended space-time symmetry, so while it might appeal to our aesthetics, SUSY is no way guaranteed by physical principles, just as homogeneity and isotropy are not necessary ingredients in quantum field theory. The fact that our universe is (to an excellent approximation) homogeneous and isotropic can either be looked at as a lucky coincidence or as the realization of some "deep fundamental principle."

We can look at this in a slightly different light by going back to extra dimensions. If the brane world proponents are correct, then the supposed translational invariance in the extra dimensions is (spontaneously?) broken by our brane. Similarly, we could have supersymmetry without ever realizing it if there was some (spontaneous?) mechanism by which supersymmetry was broken. The fact that we haven't yet seen experimental evidence for SUSY suggests that to understand supersymmetry, we have to learn not only what SUSY is and how it works, but also how a seemingly fundamental extended space-time symmetry could be broken to give us the universe we observe at low energies. Hopefully, we will have some understanding of these topics by the end of the summer.

So why SUSY in the first place? From the phenomenological point of view, we will see in subsequent talks that SUSY presents a mechanism for stabilizing the Higgs mass, so SUSY would give a natural solution to the hierarchy problem. From a numerological point of view, the beta functions in the minimally supersymmetric Standard Model (MSSM) show a unification of the gauge couplings at around 10^{16} GeV. Whether this is impressive or not depends on your opinion about GUTs. For our stringy friends, SUSY is somehow necessary/desired, so we should understand the effects of SUSY at scales much larger than the string scale. Finally, from an aesthetic point of view, the Haag-Lopuszanski-Sohnius extension of the Coleman-Mandula theorem tells us that the only (extended) space-time symmetries that can appear in non-trivial quantum field theories are translations, boosts/rotations, and SUSY. For some reason, we physicists take great pride in saying "...and that's it."

This summer, one of our goals is to understand SUSY and SUSY breaking. In this talk, I will review the $N = 1$, $d = 4$ supermultiplet formalism, and use it to write down the MSSM. Next time, Jason will talk more about the MSSM and SUSY breaking in order to understand how the MSSM even has a chance to describe our universe. We can easily do this phenomenologically through "soft terms" in the Lagrangian, but we should also look at some of the proposed spontaneous mechanisms. Later in the summer, Devin will talk about renormalization in supersymmetric theories. In particular, he will discuss the cancellation of UV divergences and the NSVZ beta function. Also, we should consider looking at SUSY with $N > 1$ in $d > 4$ to connect with some of the more exotic SUSY theories.

2 The Symmetry of Supersymmetry

As Nima remarked in his introduction to Supersymmetry, SUSY is not so dissimilar from usual space-time translation symmetry. If we have a function $f(x^\mu)$, we know that we can apply a translation using the energy-momentum four-vector:

$$e^{-ia^\mu P_\mu} f(x^\mu) = f(x^\mu + a^\mu). \tag{1}$$

Similarly, if we have a function of a fermionic (Grassman) coordinate θ, we expect that there should be a generator of fermionic translations Q such that

$$e^{\eta Q} f(\theta) = f(\theta + \eta). \tag{2}$$

(Note the lack of a factor of i in the exponent. Based on the definition of Q later, this will correspond to my notion that $e^{\eta \frac{\partial}{\partial \theta}} f(\theta) = f(\theta + \eta)$ in analogy with $e^{a \frac{\partial}{\partial x}} f(x) = f(x + a)$.) More generally, we can think of θ, η, and Q as Weyl spinors, and the above equation reads

$$e^{\eta^\alpha Q_\alpha} f(\theta_\alpha) = f(\theta_\alpha + \eta_\alpha), \tag{3}$$

where we raise and lower Weyl indices with respect to the epsilon tensor.

What if we have a function of both bosonic (x^μ) and fermionic (θ_α, $\bar{\theta}_{\dot{\beta}}$) coordinates? We could certainly think about translating this function in the fermionic direction via

$$e^{\eta^\alpha Q_\alpha + \bar{Q}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}}} f(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\beta}}) \stackrel{?}{=} f(x^\mu, \theta_\alpha + \eta_\alpha, \bar{\theta}_{\dot{\beta}} + \bar{\eta}_{\dot{\beta}}). \tag{4}$$

For this to work we would have to assume the anti-commutation relation $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 0$. But this is not the most general anti-commutation relation consistent with Lorentz covariance. Using the sigma matrices, we could have

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \tag{5}$$

with the factor of 2 set by convention. In fact, this gives us the SUSY ($N = 1$) algebra:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \quad [Q_\alpha, P_\mu] = [\bar{Q}_\alpha, P_\mu] = 0, \quad [P_\mu, P_\nu] = 0. \tag{6}$$

(More generally, we could have a set of anti-commuting supercharges $Q^A, \bar{Q}^A$ where A runs from 1 to k. This would be an $N = k$ superalgebra.)

Of course, an algebra isn't much use if we don't have a concrete representation of it. Try

$$P_\mu = i\partial_\mu, \quad Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu, \quad \bar{Q}_{\dot{\beta}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\beta}}} + i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu. \tag{7}$$

Now applying our translation in the fermionic direction (dropping Weyl indices and specializing to the limit of infinitesimal η and $\bar{\eta}$):

$$e^{\eta Q + \bar{Q}\bar{\eta}} f(x^\mu, \theta, \bar{\theta}) = e^{\eta \frac{\partial}{\partial \theta} - i\eta\sigma^\mu \bar{\theta}\partial_\mu - \frac{\partial}{\partial \bar{\theta}}\bar{\eta} + i\theta\sigma^\mu \bar{\eta}\partial_\mu} f(x^\mu, \theta, \bar{\theta}) = f(x^\mu - i\eta\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\eta}, \theta + \eta, \bar{\theta} + \bar{\eta}) \tag{8}$$

This is the infinitesimal translation we saw before:

$$x^\mu \to x^\mu - i\eta\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\eta}, \quad \theta \to \theta + \eta, \quad \bar{\theta} \to \bar{\theta} + \bar{\eta}. \tag{9}$$

Then again, this is not the only representation. We could try the left chiral representation

$$P_\mu = i\partial_\mu, \quad Q^L_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \bar{Q}^L_{\dot\beta} = -\frac{\partial}{\partial\bar{\theta}^{\dot\beta}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\partial_\mu, \tag{10}$$

under which our fermionic translation becomes

$$f_L(x^\mu, \theta, \bar{\theta}) \to f_L(x^\mu + 2i\theta\sigma^\mu\bar{\eta}, \theta + \eta, \bar{\theta} + \bar{\eta}). \tag{11}$$

Or we could try the right chiral representation

$$P_\mu = i\partial_\mu, \quad Q^R_\alpha = \frac{\partial}{\partial\theta^\alpha} - 2i\sigma^\mu_{\alpha\dot\beta}\bar{\theta}^{\dot\beta}\partial_\mu, \quad \bar{Q}^R_{\dot\beta} = -\frac{\partial}{\partial\bar{\theta}^{\dot\beta}}, \tag{12}$$

under which our fermionic translation becomes

$$f_R(x^\mu, \theta, \bar{\theta}) \to f_R(x^\mu - 2i\eta\sigma^\mu\bar{\theta}, \theta + \eta, \bar{\theta} + \bar{\eta}). \tag{13}$$

For reference, the connection between the different chiral representations are

$$f(x^\mu, \theta, \bar{\theta}) = f_L(x^\mu + i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta}) = f_R(x^\mu - i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta}). \tag{14}$$

To see this, apply the operator $e^{\eta Q + \bar{Q}\bar{\eta}}$ in the standard representation to each side of this equation.

3 Chiral Multiplets

The reason why the chiral representations are useful, is that they allow us to define functions that are independent of either θ or $\bar{\theta}$. In the standard representation, the SUSY translation $e^{\eta Q + \bar{Q}\bar{\eta}}$ mixes x^μ with both θ and $\bar{\theta}$. But in the left (right) chiral representation, the SUSY translation only mixes x^μ with θ ($\bar{\theta}$). Therefore, it is perfectly natural to define $\Phi_L(x, \theta)$ to be independent of $\bar{\theta}$ and $\Phi_R(x, \bar{\theta})$ to be independent of θ.

At this point, it is customary to expand Φ_L in terms of component fields. Because θ_α is anticommuting and α runs from 1 to 2, the most general expansion is

$$\Phi_L(x^\mu, \theta_\alpha) = \phi(x^\mu) + \sqrt{2}\theta^\alpha\psi_\alpha(x^\mu) + \theta^\alpha\theta_\alpha F(x^\mu), \tag{15}$$

where the factor of $\sqrt{2}$ is set by convention. So the multiplet Φ_L contains a complex scalar field ϕ, a Weyl spinor ψ_α, and an auxiliary field F.

How does our fermionic translation (hereafter called a SUSY transformation) act on the component fields of Φ_L? We expect that a SUSY transformation of a chiral multiplet should yield another chiral multiplet. In particular, for an infinitesimal transformation:

$$\begin{aligned}
\delta\Phi_L(x^\mu, \theta_\alpha) &= \Phi_L(x^\mu + 2i\theta\sigma^\mu\bar{\eta}, \theta_\alpha + \eta_\alpha) - \Phi_L(x^\mu, \theta_\alpha) \\
&= \phi(x^\mu + 2i\theta\sigma^\mu\bar{\eta}) + \sqrt{2}(\theta^\alpha + \eta^\alpha)\psi_\alpha(x^\mu + 2i\theta\sigma^\mu\bar{\eta}) + (\theta^\alpha + \eta^\alpha)(\theta_\alpha + \eta_\alpha)F(x^\mu + 2i\theta\sigma^\mu\bar{\eta}) \\
&\quad - \phi(x^\mu) - \sqrt{2}\theta^\alpha\psi_\alpha(x^\mu) - \theta^\alpha\theta_\alpha F(x^\mu) \\
&= \sqrt{2}\eta\psi(x) + \sqrt{2}\theta(i\sqrt{2}\sigma^\mu\bar{\eta}\partial_\mu\phi(x) + \sqrt{2}\eta F(x)) + \theta\theta(-i\sqrt{2}\partial_\mu\psi(x)\sigma^\mu\bar{\eta}).
\end{aligned} \tag{16}$$

So we see that under a SUSY transformation, the component fields transform as

$$\begin{aligned}\delta\phi &= \sqrt{2}\eta\psi, \\ \delta\psi &= \sqrt{2}\eta F + i\sqrt{2}\sigma^\mu\bar{\eta}\partial_\mu\phi, \\ \delta F &= -i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\eta}. \end{aligned} \tag{17}$$

In component field language, we see that a SUSY transformation mixes bosonic and fermionic fields. In other words, what looks like a space-time translation in superfield language becomes a symmetry that links fields with different statistics. This is the power of SUSY. Note also that the field F transforms as a total derivative under SUSY. We'll need this later on when constructing a SUSY Lagrangian.

Note the product of two left chiral multiplets is another left chiral multiplet. We can see this easily by direct multiplication, or by noting that a function of x^μ and θ multiplied by another function of x^μ and θ is yet another function of x^μ and θ.

Finally, it will be convenient to have SUSY-covariant derivatives at our disposal. In the standard representation:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{D}_{\dot{\beta}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu. \tag{18}$$

In the left chiral representation:

$$D^L_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{D}^L_{\dot{\beta}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}, \tag{19}$$

and similarly for the right chiral representation. Note that

$$\{D_\alpha, Q_\alpha\} = \{\bar{D}_{\dot{\alpha}}, Q_\alpha\} = \{D_\alpha, \bar{Q}_{\dot{\alpha}}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\alpha}}\} = 0, \tag{20}$$

which justifies the name "SUSY-covariant derivative." We can then *define* left and right chiral fields by

$$\bar{D}\Phi_L = 0, \tag{21}$$

$$D\Phi_R = 0. \tag{22}$$

Don't confuse left and right chiral fields with left and right chiral representations. Certainly, a left chiral field takes on a special form in the left chiral representation (*i.e.* $\bar{D}^L\Phi_L = 0$ implies that Φ_L is independent of $\bar{\theta}$). But we can certainly have a left chiral field expressed in the right chiral representation if it satisfies $\bar{D}^R\Phi_L^{(R)}(x,\theta,\bar{\theta}) = 0$. In this representation, Φ_L doesn't have any special properties.

4 Vector Multiplets

The chiral condition was one kind of constraint we could put on a superfield. The constraint that defines the vector multiplet is self-conjugation:

$$V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta}). \tag{23}$$

We can again expand V in components. The terms will have up to 2 θs and up to 2 $\bar{\theta}$s. The expansion I will use looks very contrived at first, but it will become useful when we talk about gauge invariance:

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\ &+ \frac{i}{2}\theta\theta\left(M(x)+iN(x)\right) - \frac{i}{2}\bar{\theta}\bar{\theta}\left(M(x)-iN(x)\right) \\ &+ \bar{\theta}\bar{\sigma}^\mu\theta A_\mu(x) + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) \\ &- i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \frac{1}{2}\Box C(x)\right). \end{aligned} \tag{24}$$

For this to be self-conjugate, C, D, M, N, and A_μ must be real functions.

Again, the reason to use this expansion is gauge transformations. A gauge transformation in ordinary field theory is

$$\begin{aligned} \phi(x) &\to e^{iq\alpha(x)}\phi(x), \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\alpha(x), \end{aligned} \tag{25}$$

where $\alpha(x)$ is a real function. We want to duplicate this in superspace. If we multiply a left chiral multiplet by a real function of x, then we no longer have a left chiral multiplet because $\bar{D}\Phi \neq 0$. Therefore, a gauge transformation on a chiral multiplet should look like

$$\Phi \to e^{q\Lambda}\Phi, \tag{26}$$

where Λ is another chiral multiplet.

What about for the vector multiplet? We're going to need to do some mysterious algebra to figure this out. First, let's write down a left chiral multiplet using the ordinary representation of the SUSY algebra. By equation (14):

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \Phi_L(x^\mu + i\theta\sigma^\mu\bar{\theta},\theta,\bar{\theta}) \\ &= \phi(x^\mu + i\theta\sigma^\mu\bar{\theta}) + \sqrt{2}\theta^\alpha\psi_\alpha(x^\mu + i\theta\sigma^\mu\bar{\theta}) + \theta^\alpha\theta_\alpha F(x^\mu + i\theta\sigma^\mu\bar{\theta}) \\ &= \phi(x^\mu) - i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi(x^\mu) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x^\mu) + \sqrt{2}\theta\psi(x^\mu) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x^\mu) + \theta\theta F(x^\mu). \end{aligned} \tag{27}$$

Now consider the expression $\Phi + \Phi^\dagger$ (I told you this would seem mysterious):

$$\begin{aligned} \Phi + \Phi^\dagger &= \phi + \phi^* + \sqrt{2}(\theta\psi + \bar{\theta}\bar{\psi}) + \theta\theta F + \bar{\theta}\bar{\theta}F^* - i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu(\phi - \phi^*) \\ &+ \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box(\phi+\phi^*). \end{aligned} \tag{28}$$

Recall that the $\bar{\theta}\bar{\sigma}^\mu\theta$ component of V was A_μ, and here we see that the $\theta\sigma^\mu\bar{\theta}$ component of $\Phi + \Phi^\dagger$ is $-i(\phi - \phi^*)$ which is a real function. Renaming Φ to Λ, this suggests that a gauge transformation on a vector multiplet should be

$$V \to V - \Lambda - \Lambda^\dagger, \tag{29}$$

for some chiral multiplet Λ. In particular, under this gauge transformation:

$$\begin{aligned} C &\to C - \phi - \phi*, \\ \chi &\to \chi + i\sqrt{2}\psi, \end{aligned}$$

$$\begin{aligned} M+iN &\rightarrow M+iN+2iF, \\ A_\mu &\rightarrow A_\mu + i\partial_\mu(\phi-\phi^*), \\ \lambda &\rightarrow \lambda, \\ D &\rightarrow D. \end{aligned} \tag{30}$$

We see that we can pick a particular gauge (Wess-Zumino gauge) by choosing $\mathrm{Re}(\phi)$, ψ, F appropriately to set C, M, N, and χ to zero. Note that in W-Z gauge, we still have the gauge freedom of $\mathrm{Im}(\phi)$ to maintain the standard gauge symmetry $A_\mu \rightarrow A_\mu - \partial_\mu\alpha$. In W-Z gauge, the component field expansion for V is:

$$V(x,\theta,\bar\theta) = \bar\theta\bar\sigma^\mu\theta A_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x). \tag{31}$$

Here, A_μ is a gauge field, λ is a Weyl spinor (called the gaugino), and D is an auxiliary field.

We could again apply the SUSY transformation to V and we would find that, as expected, D transforms as a total derivative. We will use this fact when constructing our SUSY Lagrangian. For reference:

$$\begin{aligned} \delta A_\mu &= i\eta\sigma_\mu\bar\lambda + i\bar\eta\bar\sigma_\mu\lambda, \\ \delta\lambda &= i\eta D + \sigma^{\mu\nu}\eta F_{\mu\nu}, \\ \delta D &= \bar\eta\bar\sigma^\mu\partial_\mu\lambda - \eta\sigma^\mu\partial_\mu\bar\lambda \end{aligned} \tag{32}$$

where $\sigma^{\mu\nu} = \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (Don't confuse $F_{\mu\nu}$ with the auxiliary field F, and don't confuse the SUSY-covariant derivative D_α with the auxiliary field D.)

Note if we have a left chiral multiplet Φ, then the product $\Phi^\dagger\Phi$ and the sum $\Phi+\Phi^\dagger$ are vector multiplets because they are self-conjugate. Similarly, we might ask whether it is possible to go the other way and build a chiral multiplet out of a vector multiplet. Define the object:

$$W_\alpha = \bar D\bar D D_\alpha V. \tag{33}$$

This is a left chiral multiplet because $\bar D W_\alpha \propto \bar D^3 = 0$. Note that it carries a Weyl spinor index. You might ask why we didn't just consider the expression $Z = \bar D\bar D V$. The reason is that W_α is gauge invariant while Z is not, and we know that to get a unitary theory of a massless spin-1 field we need our Lagrangian to be gauge invariant. W_α will play the same role as $F_{\mu\nu}$ in ordinary gauge theories. (To check gauge invariance recall that $\bar D\Lambda = D\Lambda^\dagger = 0$ for Λ a left chiral multiplet.)

Finally, we can extend the discussion to non-Abelian gauge theories by defining $V = V_a T^a$ and $\Lambda = \Lambda_a T^a$, where V_a is a set of vector multiplets, Λ_a is a set of left chiral multiplets that correspond to our gauge transformation, and T^a are the generators of the relevant gauge group (in the relevant representation). If Φ is a left chiral multiplet in some representation of the gauge group, then the gauge transformation now looks like:

$$\begin{aligned} \Phi &\rightarrow e^\Lambda\Phi, \\ \Phi^\dagger &\rightarrow \Phi^\dagger e^{\Lambda^\dagger}, \\ e^V &\rightarrow e^{-\Lambda^\dagger}e^V e^{-\Lambda}. \end{aligned} \tag{34}$$

We also define

$$W_\alpha = \bar D\bar D e^{-V} D_\alpha e^V, \tag{35}$$

which has the transformation property

$$W_\alpha \rightarrow e^\Lambda W_\alpha e^{-\Lambda}. \tag{36}$$

5 SUSY Lagrangians

Now we have all the tools necessary to build SUSY invariant actions. If we define an action by

$$S = \int d^4x\; \mathcal{L}(x), \tag{37}$$

then it will be SUSY invariant if $\mathcal{L}(x)$ transforms as a total derivative under SUSY, assuming as always vanishing boundary conditions. We know of two objects in particular that transform as a total derivative under SUSY: the $\theta\theta$ component of a left chiral multiplet and the $\theta\theta\bar{\theta}\bar{\theta}$ component of a vector multiplet. This is all we'll need to construct a very generic SUSY Lagrangian.

First, we should check the mass dimensions of our various fields. If a scalar/vector field has mass dimension 1, and a Weyl fermion has mass dimension 3/2, then we can see that a chiral multiplet Φ has mass dimension 1 and the fermionic coordinates θ and $\bar{\theta}$ have mass dimension $-1/2$. (The auxiliary fields have mass dimension 2.) Plugging these into the expression for V, we see that the vector multiplet V has mass dimension 0. Also, $\int d\theta\; \theta = 1$, so $d\theta$ has mass dimension 1/2.

In effective field theory language, we only need to worry about marginal and relevant couplings. So the terms in our Lagrangian can have at most mass dimension 4. To extract the $\theta\theta$ component of a left chiral multiplet, we can merely integrate $\int d^2\theta\; \Phi$. To extract the $\theta\theta$ component of a vector multiplet, we can merely integrate $\int d^2\theta d^2\bar{\theta}\; V$. Ignoring the vector multiplet for the moment, the most generic SUSY effective Lagrangian we can create just out of chiral multiplets is:

$$\mathcal{L}(x) = \int d^2\theta d^2\bar{\theta}\; \sum_i \Phi_i^\dagger \Phi_i + \int d^2\theta\; W(\Phi_i) + h.c. \tag{38}$$

where W is a analytic polynomial function of at most degree 3. (Recall that $\Phi^\dagger\Phi$ is a vector multiplet. The analyticity of W comes form the fact that in order for the $\int d^2\theta$ term to be a left chiral multiplet, W cannot have terms proportional to $\Phi_i^\dagger$.) The function W is called the superpotential. Of course, it's not at all clear what this Lagrangian means in terms of component fields, but it should be immensely satisfying that we've written down what amounts to the most general SUSY theory of interacting scalars and Weyl fermions.

Let's expand this in component fields, using

$$W(\Phi_i) = k^i\Phi_i + \frac{1}{2}m^{ij}\Phi_i\Phi_j + \frac{1}{3}g^{ijk}\Phi_i\Phi_j\Phi_k. \tag{39}$$

Some relevant components in the standard representation:

$$\begin{aligned}
\Phi_i^\dagger\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} &= \partial_\mu\phi_i^*\partial^\mu\phi_i + \bar{\psi}_i i\bar{\sigma}^\mu\partial_\mu\psi_i + F_i^*F_i, \\
\Phi_i|_{\theta\theta} &= F_i, \\
\Phi_i\Phi_j|_{\theta\theta} &= \phi_iF_j + \phi_jF_i - \psi_i\psi_j, \\
\Phi_i\Phi_j\Phi_k|_{\theta\theta} &= \phi_i\phi_jF_k - \psi_i\psi_j\phi_k + \textit{cyclic permutations}.
\end{aligned} \tag{40}$$

This gives precisely the desired kinetic terms. Also, the coefficients m^{ij} really look like fermion mass parameters and the coefficients g^{ijk} look like Yukawa couplings! If we think of W just as a function of ϕ_i we can write the Lagrangian compactly as:

$$\mathcal{L}(x) = \partial_\mu\phi_i^*\partial^\mu\phi_i + \bar{\psi}_i i\bar{\sigma}^\mu\partial_\mu\psi_i + F_i^*F_i + \left(\frac{\partial W(\phi_i)}{\partial\phi_j}F_j - \frac{1}{2}\frac{\partial^2 W(\phi_i)}{\partial\phi_j\partial\phi_k}\psi_j\psi_k + h.c\right). \tag{41}$$

We can easily integrate out the auxiliary field through the equation of motion

$$\frac{\partial \mathcal{L}}{\partial F_j} = \partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu F_j} \tag{42}$$

because $\mathcal{L}$ is independent of $\partial_\mu F$. We find that

$$F_j^* = -\frac{\partial W(\phi_i)}{\partial \phi_j}. \tag{43}$$

Plugging in for F and F^*, our Lagrangian now reads

$$\mathcal{L}(x) = \sum_i \left(\partial_\mu \phi_i^* \partial^\mu \phi_i + \bar{\psi}_i i \bar{\sigma}^\mu \partial_\mu \psi_i\right) - \sum_j \left|\frac{\partial W(\phi_i)}{\partial \phi_j}\right|^2 - \sum_{j,k}\left(\frac{1}{2}\frac{\partial^2 W(\phi_i)}{\partial \phi_j \partial \phi_k}\psi_j \psi_k + h.c.\right). \tag{44}$$

As we saw on problem set 2, $W = \frac{1}{2}m\phi^2$ is a free theory with scalar and Weyl fermion of exactly the same mass. A theory with $W = \frac{1}{3}g\phi^3$ has massless fields with a Yukawa coupling $-g\phi\psi\psi + h.c.$ and scalar interaction $g^2(\phi^*\phi)^2$. More complicated superpotentials involving more superfields will give more complicated interactions of the component fields. The thing to remember is that the superpotential determines all of the interactions in the theory.

What if we want to construct a gauge invariant Lagrangian? By our definition of a gauge transformation, the object $\Phi^\dagger e^V \Phi$ is gauge invariant. Also, we can imagine constructing gauge invariant combinations of the chiral multiplets in the superpotential. And the object W_α is gauge invariant and a chiral multiplet, so the $\theta\theta$ component of $\mathrm{tr}(W_\alpha W^\alpha)$ can appear in our Lagrangian. So we have

$$\mathcal{L}(x) = \int d^2\theta d^2\bar{\theta} \sum_i \Phi_i^\dagger e^V \Phi_i + \int d^2\theta\, W_{G.I.}(\Phi_i) + h.c. + \int d^2\theta \frac{1}{g^2}\mathrm{tr}(W_\alpha W^\alpha) + h.c. \tag{45}$$

(Don't confuse the superpotential W with the chiral field W_α that was constructed from V. Also, I'm not completely sure about the numerical coefficient in front of $1/g^2$.) Note that W_α has mass dimension $3/2$, so terms involving higher powers of W_α won't appear in our effective Lagrangian.

Once again, we can expand this out in component fields. (We will also scale $V \to 2gV$ to correspond to generally accepted conventions.) Starting with the first term

$$\int d^2\theta d^2\bar{\theta} \Phi^\dagger e^V \Phi \to D_\mu \phi D^\mu \phi^* + \bar{\psi} i \bar{\sigma}^\mu D_\mu \psi + g\phi^* D\phi + ig\sqrt{2}(\phi^*\lambda\psi - \bar{\psi}\bar{\lambda}\phi) + F^*F, \tag{46}$$

where D_μ is the standard gauge covariant derivative $D_\mu = \partial_\mu + igA_\mu^a T^a$, $D = D_a T^a$ are the auxiliary fields for V, and $\lambda = \lambda_a T^a$ are the gauginos. (In this silly notation, D now has three meanings: as a SUSY-covariant derivative, a gauge covariant derivative, and an auxiliary field.) These are exactly the matter kinetic terms for a minimal coupled gauge theory. Note that for these terms to make sense, ϕ and ψ must transform in the same representation of the gauge group. This will place constraints on the superpartners in the MSSM. Also note that the couplings to the gauginos go as the gauge coupling.

The other new term in the gauge invariant Lagrangian is

$$\int d^2\theta \frac{1}{g^2}\mathrm{tr}(W_\alpha W^\alpha) + h.c. \to -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \frac{1}{2}D^a D_a + \bar{\lambda}^a i \bar{\sigma}^\mu D_\mu \lambda_a, \tag{47}$$

where $F_{\mu\nu}^a$ is the same one from ordinary non-Abelian gauge theory. Again, we have exactly the boson kinetic terms necessary for unitarity. Note that λ transforms in the same representation as the gauge bosons (namely, the adjoint representation), so $D_\mu \lambda_a = \partial_\mu \lambda^a - g f_{abc} A_\mu^b \lambda^c$. Again, we could integrate out the auxiliary fields F and D, and we would find coupling between the matter fields that go as the gauge coupling without actually being mediated by a gauge boson or gaugino.

6 The Minimal Supersymmetric Standard Model

Now that we have constructed a general gauge invariant Lagrangian, we can easily write down the MSSM. In superfield language, the MSSM looks like a straightforward extension of the $\{q\ u^c\ d^c\ l\ e^c\}$ Standard Model. In fact, just as we could uniquely define the Standard Model in terms of the gauge group representations of the matter fields, we can (almost) uniquely define the MSSM in terms of the the gauge group representations of the superfields. The only real difference is that we will have to introduce another Higgs doublet. Also, in later talks, we will need to introduce "soft terms" which characterize the SUSY breaking in order to give a phenomenologically valid model.

The $\Phi^\dagger e^V \Phi$ terms in the Lagrangian are set by the gauge representations of the chiral multiplets, and the $\mathrm{tr}(W_\alpha W^\alpha)$ terms are set by the gauge groups themselves, so in order to define the MSSM, we need only write down the group representations of the superfields and the superpotential. Here it is:

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
Q_i	3	2	$+1/6$
U_i^c	$\bar{3}$	—	$-2/3$
D_i^c	$\bar{3}$	—	$+1/3$
L_i	—	2	$-1/2$
E_i^c	—	—	1
H	—	2	$-1/2$
$\bar{H}$	—	2	$+1/2$

$$W_{MSSM} = \mu H\bar{H} + \sum_{i,j} \left(\lambda_{ij}^E H L_i E_j^c + \lambda_{ij}^D H Q_i D_j^c + \lambda_{ij}^U \bar{H} Q_i U_j^c \right) \tag{48}$$

where i runs from 1 to 3 and labels the lepto-quark generations, and I have suppressed $SU(2)$ and $SU(3)$ indices. (Insert epsilon tensors as needed.) Note that there are two higgs doublets, H and $\bar{H}$. In order to have interactions between Q and U^c, we needed to have an $SU(2)$ doublet with hypercharge $+1/2$. We might have thought that we could use $H^\dagger$ (the 2 of $SU(2)$ is the same as the $\bar{2}$), but we saw before that W had to be an analytic function of its arguments in order for it to remain a chiral multiplet. So we have to add the $\bar{H}$ field in order to give the up quark a mass after symmetry breaking.

We know that none of the Standard Model fermions can be gauginos because none of the matter fields transform in the adjoint representation. Still, looking at the table, how do we know that the slepton of the L field isn't just the standard Higgs doublet? It turns out that in order for this to work, we would have to have interactions that break lepton number. Furthermore, the neutrinos would acquire dangerously large masses, way beyond the experimental bounds. So a two Higgs doublet with the above superpotential is indeed the MSSM.

However, this superpotential is *not* uniquely determined in the same way as the interactions in the Standard Model were uniquely determined. In the Standard Model, terms in the Lagrangian like $qd^c l$ are excluded because they are irrelevant operators. But in the MSSM, terms in the superpotential of the form

QD^cL cannot be discarded by dimensional analysis because we can have terms with up to three superfields! Just looking at gauge invariance, we could have the follow terms:

$$W_? = QD^cL + LLE^c + HHE^c + \bar{H}L + U^cD^cD^c \tag{49}$$

(We've used the epsilon tensor in the last term.) These operators break baryon number and lepton number, which were accidental symmetries in the standard model. Do we have any good reasons for discarding these terms, or do we have to wave our hands to make them disappear? (The answer is R parity.) Also, what does electroweak symmetry breaking look like in the MSSM? (The answer is that we need to break SUSY to break electroweak.) These and other issues we'll study in the next episode of... *Beyond the Standard Model*.

7 Looking Ahead

Now that we have outlined some of the basics of SUSY, we need to step back and ask ourselves what we've accomplished. Certainly, the aesthetics of SUSY is appealing; it is quite easy to write SUSY invariant Lagrangians in the superfield formalism. And as we'll see later in the summer, SUSY handles the quadratic divergences at the heart of the hierarchy problem by presenting a link between bosons and fermions. But returning to the question I asked at the beginning of this talk: do we have any reason to expect that our universe is supersymmetric?

Certainly, by the fact that we don't observe supersymmetric particles, we know that our universe has a non-supersymmetric vacuum state. Thus, SUSY has to be spontaneously broken in order to have even a chance at describing our universe. Later this summer, we'll learn more about SUSY breaking and how it might be accomplished.

For some physicists, the fact that SUSY controls quadratic divergences is reason enough to expect it might be part of theories beyond the Standard Model. But we saw that in either the ADD or RS1 scenarios, the hierarchy problem can be solved using extra dimensions (without needing to reference SUSY) by postulating that the weak scale (or some other energy scale less than $\sim$10 TeV) *is* the fundamental mass scale, so the fine tuning of the Higgs mass is no longer a problem. Even more interesting, theory space models (like the one Rakhi and Can are studying) can control quadratic divergences by introducing partner particles with *the same* statistics. Therefore, SUSY's raison d'être will have to go beyond just the hierarchy problem if we want to believe that SUSY is a true symmetry of our universe.

References

[1] Manuel Drees. *An Introduction to Supersymmetry.* hep-ph/9611409

This is the basis for much of these notes. Quick and easy, for the most part. Just so you know, almost every SUSY introduction uses different notation. There are at least three different gauge coupling conventions and an equal number of different fermion types used. I highly recommend learning SUSY using superfield notation with Weyl fermions to start. Apparently, superfields aren't that useful later on, but all the introductions that I found that didn't use superfields had to do a lot of handwaving to justify the form of the SUSY transformations on component fields. (And using Majorana fermions is just plain frustrating.) In any case, now that we know the basics of SUSY, the interesting articles for us will be ones that move away from the formalism and discuss how to make contact with observed phenomena.

[2] Stephen Martin. *A Supersymmetry Primer.* hep-ph/9709356

A great introduction to the phenomenology of SUSY. Unfortunately, he doesn't use the superspace formalism which I find quite enlightening. He does use Weyl fermions, thankfully.

[3] Nir Polonsky. *Supersymmetry: Structure and Phenomena.* hep-ph/0108236

From what I can tell from the table of contents, this introduction to SUSY is packed with experimental consequences. I found the discussion of the SUSY formalism to be a bit too brief, but this paper should be helpful when looking at matching SUSY to the Standard Model.

[4] Philip Argyres. *Introduction to Global Supersymmetry.* http://www.physics.uc.edu/∼argyres/661/

These lectures are a good introduction to SUSY, though the point of view taken is slightly different than the one we have been following in the reading course: SUSY is introduced first in the context of Quantum Mechanics (*i.e.* $0+1$ dimensional field theory). Note that Argyres insists on using Majorana fermions as opposed to Weyl fermions.

[5] Julius Wess and Jonathan Bagger. *Supersymmetry and Supergravity.* Princeton University Press, 1992.

I found this text difficult to follow. Lots of equations, not much explanation. A good reference for the formal aspects of SUSY.