Appendix to Gauged Ghost Paper

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A Unhealthy Modifications to Gravity

The theories of ghost condensation and gauged ghost condensation suggest a new route for trying to find modifications of gravity. In both cases, the low energy effective Lagrangian can be obtained by considering the most general Lagrangian consistent with whichever unbroken gauge symmetries remain. This suggests that we could construct a whole class of modifications of gravity simply by postulating some reduced diffeomorphism symmetry and finding general Lagrangians consistent with the unbroken diffeomorphisms.

In this appendix, we show two examples where this reasoning leads to unhealthy (but Lorentz-invariant) modifications of gravity. In both cases, we can make choices to reduce the theory to (healthy) scalar-tensor theories. However, these choices simply amount to eliminating any couplings between manifestly sick modes and external sources. Despite the fact that these theories are sick, they serve to remind us that by breaking diffeomorphisms, we are simply introducing new degrees of freedom. Whether these new degrees of freedom are interesting or not depends on the specifics of the breaking pattern. In the case of ghost condensation or gauged ghost condensation we find a fascinating Goldstone mode with a Lorentz-violating dispersion relation. In the two examples in this appendix, we find nothing but violations of unitarity.

How many ways can we break diffeomorphisms in a Lorentz-invariant way? An arbitrary diffeomorphism ξ^μ can be decomposed into its "longitudinal" and "transverse" pieces:

$$\xi^\mu = \partial^\mu \theta + \hat{\xi}^\mu, \qquad \partial_\mu \hat{\xi}^\mu = 0. \tag{1}$$

The case of breaking all diffeomorphisms corresponds to massive gravity (see [ref?]). Clearly, we can also separately break longitudinal or transverse diffeomorphism. Under θ and $\hat{\xi}^\mu$, the symmetric tensor $h_{\mu\nu}$ transforms as:

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} - 2\partial_\mu \partial_\nu \theta, \\ h_{\mu\nu} &\rightarrow h_{\mu\nu} - \partial_\mu \hat{\xi}_\nu - \partial_\nu \hat{\xi}_\mu. \end{aligned} \tag{2}$$

The two examples in this appendix are the two Lorentz-invariant possibilities of breaking θ or breaking $\hat{\xi}$.

A.1 One New Degree of Freedom

We will start with the case of breaking longitudinal diffeomorphisms θ. In that case, we expect one new scalar degree of freedom. The leading two-derivative quadratic Lagrangian we can write down for $h_{\mu\nu}$ consistent with $\hat{\xi}^\mu$ is:

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R + \beta\epsilon h R + \frac{\epsilon^2}{2}(\partial_\mu h)^2 - \frac{\epsilon^2 m^2}{2} h^2 \right) + h_{\mu\nu}T^{\mu\nu}, \tag{3}$$

where β, ϵ, and m^2 are arbitrary parameters, $h = h^\mu_\mu$, and to leading order

$$\begin{aligned} \sqrt{g}R &= \frac{1}{2}(\partial_\omega h_{\mu\nu})^2 + (\partial^\mu h)(\partial^\nu h_{\mu\nu}) - \frac{1}{2}(\partial_\mu h)^2 - (\partial^\nu h_{\mu\nu})^2, \\ R &= \partial^\mu\partial^\nu h_{\mu\nu} - \Box h. \end{aligned} \tag{4}$$

Note that in order for the source term $h_{\mu\nu}T^{\mu\nu}$ to be invariant under the remaining transverse diffeomorphisms, $T_{\mu\nu}$ must satisfy (up to total derivatives)

$$\hat{\xi}_\mu \partial_\nu T^{\mu\nu} = 0 \qquad \Longrightarrow \qquad \partial_\nu \left(T^{\mu\nu} - \frac{1}{\Box}\partial_\omega\partial^\mu T^{\nu\omega} \right) = 0. \tag{5}$$

Here we have used the fact that we can formally write $\hat{\xi}_\mu$ as $\xi_\mu - \frac{1}{\Box}\partial_\mu(\partial\cdot\xi)$.

To isolate the new degree of freedom, we can perform the broken θ diffeomorphism and promote it to a field φ.

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R + \beta\epsilon(h + 2\Box\varphi)R + \frac{\epsilon^2}{2}(\partial_\mu h + 2\partial_\mu\Box\varphi)^2 - \frac{m^2\epsilon^2}{2}(h + 2\Box\varphi)^2 \right) + (h_{\mu\nu} + 2\partial_\mu\partial_\nu\varphi)T^{\mu\nu}. \tag{6}$$

Already, this theory looks peculiar because there is no normal $(\partial_\mu\varphi)^2$ kinetic term. To make the physics more transparent, we can make a field redefinition on φ:

$$\tilde{\varphi} = \epsilon(h + 2\Box\varphi). \tag{7}$$

The Lagrangian in terms of φ is

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R + \beta\tilde{\varphi}R + \frac{1}{2}(\partial_\mu\tilde{\varphi})^2 - \frac{m^2}{2}\tilde{\varphi}^2 \right) + h_{\mu\nu}\tilde{T}^{\mu\nu} + \frac{1}{\epsilon}\tilde{\varphi}\chi, \tag{8}$$

where

$$\chi = \frac{1}{\Box}\partial_\mu\partial_\nu T^{\mu\nu}, \qquad \tilde{T}^{\mu\nu} = T^{\mu\nu} - \eta^{\mu\nu}\chi. \tag{9}$$

For $\epsilon^2 > 0$ and $m^2 > 0$, this looks like a theory of a scalar $\tilde{\varphi}$ with a healthy kinetic term coupled to gravity with sources. Note that by equation (5), $\partial_\mu\tilde{T}^{\mu\nu} = 0$, so $h_{\mu\nu}$ couples to a conserved current as demanded by full diffeomorphism invariance.

However, the sources χ and $\tilde{T}^{\mu\nu}$ are manifestly non-local, so a local source $T^{\mu\nu}$ will generate unbounded energy configurations in $h_{\mu\nu}$ and φ. We can cure these instabilities by forcing $\partial_\mu T^{\mu\nu} = 0$, but then the theory simply reduces to a scalar-tensor theory of gravity.

A.2 Three New Degrees of Freedom

We now turn to the case of breaking transverse diffeomorphisms $\hat{\xi}$. In that case we expect three new propagating degrees of freedom. Again, we will see that we can write the Lagrangian in terms of fields with normal kinetic terms, but only if we have non-local sources. The leading Lagrangian consistent with the unbroken θ diffeomorphism is

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R - \frac{\alpha^2}{2}(\partial^\nu h_{\mu\nu} - \partial_\mu h)^2 \right) + h_{\mu\nu}T^{\mu\nu}. \tag{10}$$

In order for the source term $h_{\mu\nu}T^{\mu\nu}$ to be invariant under the remaining longitudinal diffeomorphisms, $T_{\mu\nu}$ must satisfy

$$\partial_\mu \partial_\nu T^{\mu\nu} = 0. \tag{11}$$

To isolate the new degrees of freedom, we perform the broken diffeomorphism $\hat{\xi}^\mu$ and promote it to a field $\hat{A}^\mu$ with the constraint $\partial^\mu \hat{A}_\mu = 0$. Equivalently, we can take

$$\hat{A}_\mu \equiv A_\mu - \frac{1}{\Box}\partial_\mu(\partial \cdot A). \tag{12}$$

In terms of A_μ the Lagrangian is

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R - \frac{\alpha^2}{2}\left(\psi_\mu + \Box A_\mu - \partial_\mu(d\cdot A)\right)^2 \right) + \left(h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu\right)T^{\mu\nu}, \tag{13}$$

where we have used the conservation condition in equation (11), and

$$\psi_\mu = \partial^\nu h_{\mu\nu} - \partial_\mu h, \qquad \partial^\mu \psi_\mu = R. \tag{14}$$

While it is possible to do field redefinitions to write this Lagrangian in a more transparent way, a simpler method to understand the physics is to simply integrate out the field A_μ. The A_μ equation of motion is

$$M_{\rm Pl}^2 \alpha^2 \left(\psi^\mu + \Box A^\mu - \partial^\mu(d\cdot A) - \frac{1}{\Box}\partial^\mu R \right) + \frac{1}{\Box}\partial_\nu T^{\mu\nu} = 0. \tag{15}$$

Integrating out A_μ from equation (13):

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R + \frac{\alpha^2}{2}R\frac{1}{\Box}R \right) + h_{\mu\nu}\tilde{T}^{\mu\nu} + \frac{2}{M_{\rm Pl}^2\alpha^2}J_\mu \frac{\eta^{\mu\nu}}{\Box^2}J_\nu, \tag{16}$$

where

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{\Box}\partial^\mu J^\nu - \frac{1}{\Box}\partial^\nu J^\mu, \qquad J^\mu = \partial_\nu T^{\mu\nu}. \tag{17}$$

Note that $\partial_\mu J^\mu = 0$, so $h_{\mu\nu}$ couples to a conserved current. We can equivalently write this Lagrangian in terms of new fields $\tilde{\varphi}$ and $\tilde{A}_\mu$:

$$\mathcal{L} = M_{\rm Pl}^2 \left(\sqrt{g}R + \alpha\tilde{\varphi}R + \frac{1}{2}(\partial_\mu\tilde{\varphi})^2 - \frac{1}{4g^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \right) + h_{\mu\nu}\tilde{T}^{\mu\nu} + \tilde{A}_\mu \frac{1}{\sqrt{-\Box}}J_\mu, \tag{18}$$

where $g = 2/\alpha$ and $\tilde{F}_{\mu\nu}$ is the Faraday tensor for $\tilde{A}_\mu$. Obviously, equation (16) can be obtained from equation (18) by integrating out $\tilde{\varphi}$ and $\tilde{A}_\mu$. Also, it is clear we need $\alpha^2 > 0$ and we need to include appropriate gauge fixings for both $h_{\mu\nu}$ and $\tilde{A}_\mu$.

This theory has some remarkable features, including a relation between the gauge coupling $g = 2/\alpha$ and the $\tilde{\varphi}R$ coupling α that is *guaranteed* by the original θ diffeomorphism symmetry. However, we see that $h_{\mu\nu}$ and $\tilde{A}_\mu$ both couple to non-local currents, and therefore this theory has drastic instabilities. If we made the choice $\partial_\mu J^\mu = 0$, then these instabilities would be cured, but we would be left simply with a scalar-tensor theory with a decoupled massless spin-1 field.

A.3 What Went Wrong

In the case of massive gravity, we break all diffeomorphism symmetry but at tree level we only allow the inclusion of mass terms for $h_{\mu\nu}$ and not new two-derivative kinetic terms. In the first example we saw, the residual transverse diffeomorphism symmetry only allowed an h^2 mass term, but that term alone only partial gauge-fixes to traceless gauge. In the second example, the residual longitudinal diffeomorphism forbid any mass term, so the only way to get any modification of gravity in these examples was to modify the $\sqrt{g}R$ kinetic terms. But by modifying the two-derivative pieces, we saw that the new modes have couplings to non-local sources.

The general Lorentz-invariant propagator for a symmetric tensor field $h_{\mu\nu}$ when there are no mass terms or new physical scales is

$$\Delta^{\alpha\beta\mu\nu}(p) = \frac{1}{p^2}\left(\lambda_1 \eta^{\alpha\mu}\eta^{\beta\nu} + \lambda_2 \eta^{\alpha\beta}\eta^{\mu\nu} + \lambda_3 \eta^{\alpha\beta}\frac{p^\mu p^\nu}{p^2} + \lambda_4 \eta^{\alpha\mu}\frac{p^\beta p^\nu}{p^2} + \lambda_5 \frac{p^\alpha p^\beta p^\mu p^\nu}{p^4} + \text{sym.}\right), \qquad (19)$$

where we symmetrize $\alpha \leftrightarrow \beta$, $\mu \leftrightarrow \nu$, and $\alpha\beta \leftrightarrow \mu\nu$. If $h_{\mu\nu}$ couples to a source that does not satisfy $\partial_\mu T^{\mu\nu} = 0$, then the dangerous $1/p^4$ and $1/p^6$ poles in the propagator are not protected, and the theory propagates ghost modes. Classically, a point source sets up a $h_{\mu\nu}$ field that grows as r or r^3, and the essential assumption that fields die off at the boundary of space-time fails.

Therefore, without mass terms (and assuming Lorentz-invariance) the only allowed modifications of gravity are ones where energy-momentum tensors are still conserved but λ_1 and λ_2 are modified from their GR values $\lambda_1 = +1/2$ and $\lambda_2 = -1/2$ (of course the other λ_i can also change, but this won't change any of the physics). Modifying λ_1 just amounts to rescaling fields. Modifying λ_2 effectively mixes $h_{\mu\nu}$ with a new scalar mode, so the only massless, Lorentz-invariant modification of gravity is scalar-tensor theory. Of course, current experimental bounds suggest that λ_1 and λ_2 are very close to their GR values, and while scalar-tensor theory is not excluded, the coupling of the scalar to R must be very weak.

What massive gravity, ghost condensation, gauged ghost condensation, DGP gravity, etc. tell us is that if we include 4D effective mass terms or break Lorentz-invariance, we can safely modify gravity over some energy range. In all these known cases, the low energy 4D description needs a UV completion below $M_{\rm Pl}$. The two examples in this appendix simply emphasize that ghost condensation and gauged ghost condensation are special cases where simplying reducing the diffeomorphism symmetry gives generically healthy theories over some energy range.

B Gauge Fixing Gravity

In the course of looking at these examples, I discovered something that is probably well-known but I've never seen before. One might ask, what are all the valid Lorentz-invariance gauge fixings for gravity that don't introduce new mass scales? Looking at equation (19), we see that a gauge fixing that touches λ_5 must have a $(\partial_\mu \partial_\nu h^{\mu\nu})^2$ term, but by dimensional analysis, the coefficient of this term must be zero or infinity in order for this term to not introduce a new scale.

The general Lagrangian we will consider is

$$\mathcal{L} = A(\partial_\omega h_{\mu\nu})^2 + B(\partial^\mu h)(\partial^\nu h_{\mu\nu}) + C(\partial_\mu h)^2 + D(\partial^\nu h_{\mu\nu})^2 + E(\partial_\mu \partial_\nu h^{\mu\nu})^2. \tag{20}$$

The (un-gauge-fixed) GR values are

$$A = \frac{1}{2}, \qquad B = 1, \qquad C = -\frac{1}{2}, \qquad D = -1, \qquad E = 0. \tag{21}$$

We want to find all values of A, B, C, and D (with $E = 0$ or ∞) such that the propagator in equation (19) still has $\lambda_1 = 1/2$, $\lambda_2 = -1/2$ and the other λ_i are finite. To find the propagator, we write the Lagrangian in equation (20) as

$$\mathcal{L} = -\frac{1}{2} h_{\alpha\beta} M^{\alpha\beta\mu\nu} h_{\mu\nu} \tag{22}$$

where $M^{\alpha\beta\mu\nu}$ is appropriately symmetrized. The propagator is then the solution (in momentum space) of

$$M_{\alpha\beta\mu\nu} \Delta^{\mu\nu\rho\sigma} = \delta^\rho_\alpha \delta^\sigma_\beta. \tag{23}$$

Here are the only possibilities:

$$E = 0: \qquad A = \frac{1}{2}, \quad B = 1 - \frac{2\eta}{\xi}, \quad C = -\frac{1}{2} + \frac{\eta^2}{\xi}, \quad D = -1 + \frac{1}{\xi}$$

$$\implies \qquad \mathcal{L}_{\rm GF} = \frac{1}{\xi}(\partial^\nu h_{\mu\nu} - \eta \partial_\mu h)^2, \qquad \xi \neq \infty, \quad \eta \neq 1. \tag{24}$$

And...

$$E = \infty: \qquad A = \frac{1}{2}, \quad B = 1 + \eta, \quad C = -\frac{1}{2}, \quad D = -1 + \frac{1}{\xi}$$

$$\implies \qquad \mathcal{L}_{\rm GF} = \frac{1}{\xi}(\partial^\nu h_{\mu\nu})^2 + \eta(\partial^\mu h)(\partial^\nu h_{\mu\nu}) + \infty(\partial_\mu \partial_\nu h^{\mu\nu})^2, \qquad \xi \neq \infty, \quad \eta \text{ can be anything}. \tag{25}$$

Well there it is, for what its worth.