Phase Space of the Ghost Condensate

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1 An Opportunity to Explore Self-Consistency

The theory of ghost condensation [1] has been introduced as a consistent IR modification of gravity, valid up to an energy scale M. Though not necessary, the ghost condensate can couple directly to Standard Model fermions, and recently we have explored some of the classical, non-relativistic dynamics coming from ghost/SM interactions [2]. Because the ghost condensate has a self-consistent low energy description, we expect that S-matrix elements involving the scattering of ghost condensate excitations (hereafter called π fields) should be unitary. But because the π field has an unusual $\omega \sim k^2/M$ dispersion relation, we should also check that the cross-sections we obtain for processes involving π-ons are well-behaved.

This is not as trivial an exercise as it might sound. The cross-section involves an integral over phase space for the external states, and because the π field violates Lorentz invariance, we cannot use the Lorentz invariant phase space. We can find the correct phase space either by feigning naïveté or appealing to the optical theorem, and when we try to calculate the center-of-mass two-particle phase space for a π-on and a normal particle of mass m, we will find that the phase space diverges as $1/\sqrt{E_{\text{cm}} - m}$ as E_{cm} approaches m. This should be contrasted with the case of the two-particle phase space for a photon and an electron which goes linearly to zero as the center-of-mass energy approaches m_e.

There are many ways that this problem might be averted. In particular the π field couples like a Goldstone boson, and we might expect that the added derivative interactions could compensate for the divergent phase space in the expression for the cross-section. Unfortunately, this turns out not to be the case. Also, we note that for any reasonable system, the center-of-mass energy is never that close to a mass threshold. In particular, for $e^-\gamma \to e^-\pi$ (a pseudo-Compton scattering event), the photon always has *some* energy, so E_{cm} is strictly greater than m_e. However, the $\omega_{\text{photon}} \to 0$ limit of normal Compton scattering does have a meaningful interpretation, and it is worrisome that the same cannot be said of the pseudo-Compton effect.

In this talk, we will show that cross-sections involving π-ons are in fact well-behaved. We will accomplish this not by some theoretical trick or an arbitrary IR cut-off, but rather by appealing to the theory of ghost condensation itself. In particular, in order to add a direct coupling between the π field and Standard Model fermions, you have to modify the dispersion relations for those fermions. When we take into account this modified dispersion relation, we will find that the two-particle center-of-mass phase space for a π-on and an electron is not divergent, and that the cross-section for, say, pseudo-Compton scattering is well-behaved.

Let us emphasize just how important this point is. One might think that there was no difficulty considering a scalar field ϕ that happened to have a Lorentz-violating $\omega \sim k^2/M$ dispersion relation. Up to a scale M, this theory looks like a perfectly sensible effective field theory. In order to guarantee that the ϕ field was exactly massless, we would have to introduce a shift symmetry $\phi \to \phi + c$, so ϕ has to be derivatively coupled to SM fields. (If the ϕ field were not derivatively coupled and the ϕ field did have a mass, then the $\omega_{\text{photon}} \to 0$ limit of pseudo-Compton scattering would not be kinematically allowed, and the divergent phase space issue would be resolved.) If we then tried calculating the cross-section for pseudo-Compton scattering with the ϕ field, we would find that it diverged as the incoming photon became softer and softer, and we might be led to the conclusion that either the electron or the photon dispersion relation has to be modified. In our case, residual space diffeomorphism symmetry [3] tells us exactly how the electron dispersion relation has to be modified and naturally introduces an "IR cutoff" for our theory.

2 One Particle Phase Space for the Ghost Condensate

For a relativistic particle of mass m, the one particle phase space is

$$\int d\Pi_1 = \int \frac{d\omega\, d^3k}{(2\pi)^4}(2\pi)\delta(\omega^2 - k^2 - m^2)\theta(\omega). \tag{1}$$

It is easy to check that this phase space (including the θ function) is invariant under Lorentz transforms. Performing the ω integral,

$$\int d\Pi_1 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_m(k)} \tag{2}$$

where $E_m(k) = \sqrt{k^2 + m^2}$. This is the familiar Lorentz-invariant phase space for a particle of mass m.

We might then guess (correctly) that the one-particle phase space for a π-on is

$$\int d\Pi_1^\pi = \int \frac{d\omega\, d^3k}{(2\pi)^4}(2\pi)\delta\left(\omega^2 - \frac{k^4}{M^2}\right)\theta(\omega) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi(k)}, \tag{3}$$

where $E_\pi(k) = k^2/M$. Clearly this expression imposes the on-shell condition, but why didn't we choose the argument of the delta function to be simply $(\omega - k^2/M)$ or have additional weighting factors in the integral? There are two reasons. First, the factor of $1/2E_m$ in the relativistic one particle phase space comes from normalizing external states in the expression for the cross-section. This has everything to do with the on-shell condition for the external particle and nothing to do with Lorentz invariance, therefore it should carry over to the π-on case. Second, by the optical theorem, we want the imaginary part of the forward scattering amplitude to be related to the total cross section, and by the Cutkosky rules, the only place we can get imaginary parts for amplitudes is from propagators. If we cut an internal π-on line, we get an imaginary part

$$\operatorname{Im} \frac{1}{\omega^2 - k^4/M^2 + i\epsilon} \sim \delta\left(\omega^2 - \frac{k^4}{M^2}\right). \tag{4}$$

This is precisely the phase space "weighting" we expected, modulo the $\theta(\omega)$ factor which we could extract in principle if we were more careful.

Note that because the π-on phase space is not Lorentz invariant, the expression in equation (3) is only valid in the ether rest frame. Performing a boost along the $\hat{z}$ direction

$$\omega \to \frac{\omega + v k_z}{\sqrt{1-v^2}}, \qquad k_x \to k_x, \qquad k_y \to k_y, \qquad k_z \to \frac{k_z + v\omega}{\sqrt{1-v^2}}. \tag{5}$$

The phase space becomes (with corrections at order v^2)

$$\int d\Pi_1^\pi = \int \frac{d\omega\, d^3k}{(2\pi)^4}(2\pi)\delta\left(\omega^2 - \frac{k^4}{M^2} + 2vk_z\omega\left(1 - \frac{2k^2}{M^2}\right)\right)\theta(\omega + vk_z). \tag{6}$$

Note that a physical state that satisfied the positive energy condition in the ether rest frame might have negative energy in the boosted frame. Whether it is easier to do calculations in the ether rest frame or the center-of-mass frame is a matter of taste, but rarely would terrestrial experiments occur in the ether rest frame. In particular, if the CMB rest frame is the ether rest frame, then typically $v \sim 10^{-3}$ and we are justified in ignoring terms at order v^2.

3 Naïve Two Particle Phase Space For Electrons and π-ons

To find the two particle phase space for an electron and a π-on, we merely multiply the two one-particle phase spaces together with an energy-momentum conserving delta function. We will work in the idealized case that the center-of-mass frame is the same as the ether rest frame, though a similar analysis holds for non-zero (but small) v.

The two particle center-of-mass phase space is

$$\int d\Pi_2 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_m(p)}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi(k)}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})\,(2\pi)\delta\left(E_{\rm cm} - E_m(p) - E_\pi(k)\right). \tag{7}$$

Performing all of the integrals expect the angular integral of k:

$$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2}\frac{M^2}{k_*\left(M + 2E_m(k_*)\right)}, \tag{8}$$

where k_* is the positive solution to $E_{\rm cm} = E_m(-k) + E_\pi(k)$. This result should be compared to the result for the two particle phase space for relativistic particles:

$$\int d\Pi_2^{\rm LI} = \int \frac{d\Omega}{16\pi^2}\frac{k_*}{E_{\rm cm}}, \qquad E_{\rm cm} = E_{m_1}(-k_*) + E_{m_2}(k_*). \tag{9}$$

Unlike the relativistic case, the phase space in equation (8) diverges as k_* goes to zero. In particular,

$$k_* \sim \sqrt{\frac{2mM}{2m+M}}\sqrt{E_{\rm cm} - m} \tag{10}$$

for $E_{\rm cm}$ close to m.

Of course, just because the phase space diverges, this does not necessarily mean that cross sections diverge. In particular, the π field couples as a Goldstone boson, so the leading (non-relativistic) coupling to a Standard Model fermion such as the electron is [3]

$$\mathcal{L}_{\rm int} = \frac{1}{F}\bar{\Psi}\gamma^5\vec{\gamma}\Psi \cdot \vec{\nabla}\pi. \tag{11}$$

If k is the magnitude of the spacial momentum for the π-on in the case of pseudo-Compton scattering ($e^-\gamma \to e^-\pi$), then the amplitude goes at best (*i.e.* most convergent for small k) as

$$\mathcal{M} \sim \frac{k}{F}. \tag{12}$$

The expression for the cross section is

$$\sigma_{\rm cm} = \frac{1}{2E_A 2E_B |v_A - v_B|}\int d\Pi_2\, |\mathcal{M}|^2, \tag{13}$$

where E_A is the energy of the initial photon, E_B is the energy of the initial electron, and $|v_A - v_B|$ is the relative velocities of the two initial particles in the center-of-mass frame. In the limit of $E_{\rm cm}$ near m_e,

$$E_A \sim k_*^2 \frac{2m_e + M}{2m_e M}, \qquad E_B \sim m_e, \qquad |v_A - v_B| \sim 1, \tag{14}$$

so we see readily that the cross section is at least as divergent as

$$\sigma_{\rm cm} \sim \frac{1}{k_*^2}\int d\Omega \frac{1}{k_*}|k_*|^2 \sim \frac{1}{k_*}. \tag{15}$$

The only way that this could be zero is if the angular integral over the phase space were zero, but this is not the case. Also, this result is robust to introducing a small "ether wind" velocity v. In fact, if we actually do the calculation we see that the divergence is worse than our optimistic guess.

$$\sigma_{\rm cm} \sim \frac{e^2}{F^2}\frac{M^5 m_e^2}{(2m_e + M)^4}\frac{1}{k_*^3} \qquad (E_{\rm cm} \to m_e) \tag{16}$$

Thankfully, in the limit that $m_e = 0$, the cross section is zero as we would expect, because the interaction in equation (11) can be removed by a field redefinition when there is an additional axial $U(1)$ symmetry on the electron field.

4 What Went Wrong?

We have a situation with an (unphysical) IR divergence, and one might first think that this is the same IR divergence that occurs with any massless field. In loop calculations in QED, some amplitudes have IR divergences that can be cured by considering processes with arbitrarily soft external photons. The soft photons can be thought of as representing the Coulomb field of the asymptotic electron or as photons that are below the energy threshold of experimental detectors.

Here, however, the amplitude of pseudo-Compton scattering is *not* IR divergent. In fact, it takes on a finite value as k_* goes to zero. Our problem stems from the divergence in the phase space and in the cross-section pre-factors, not from the amplitude. There are two likely possibilities: either something went wrong in our analysis, or it is inconsistent to couple a relativistic fermion to the π field.

I argue that the second possibility is more likely for the simple reason that the theory of ghost condensation *predicts* that a coupling to the fermion axial current must be joined by a modification to the Dirac equation for the electron. In particular, by looking at the constraints coming from residual space diffeomorphisms [3], the modification to the electron Lagrangian must be

$$\Delta\mathcal{L} = \frac{1}{F}\bar{\Psi}\gamma^5\vec{\gamma}\Psi\cdot\vec{\nabla}\pi + \mu\bar{\Psi}\gamma^0\gamma^5\Psi, \tag{17}$$

where $\mu = M^2/F$. We see that this leads to the modified Dirac equation

$$(\not{p} - m + \mu\gamma^0\gamma^5)\Psi(p) = 0, \tag{18}$$

so we have no reason to expect that electrons will continue to have a relativistic $\omega = \sqrt{k^2+m^2}$ dispersion relation. In particular, our new dispersion relation is

$$\omega = \sqrt{(|\vec{p}\,| \pm \mu)^2 + m_e^2}, \tag{19}$$

where roughly speaking, the plus sign is for left-handed electrons and positrons and the minus sign is for right-handed electrons and positrons.

5 Revised Two Particle Phase Space For Electrons and π-ons

We can use this dispersion relation to recalculate the two particle phase space. To start, we will focus on the plus sign case of equation (19). The analysis for the minus sign case is similar, but because $|p| = 0$ does not represent the lowest energy state of the fermion, the analysis is slightly altered.

Following equation (7), the two particle phase space is

$$\int d\Pi_2 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\tilde{E}_m(p)}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_\pi(k)}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})\,(2\pi)\delta\left(E_{\rm cm} - \tilde{E}_m(p) - E_\pi(k)\right), \tag{20}$$

where $\tilde{E}_m(p) = \sqrt{(|\vec{p}\,|+\mu)^2 + m_e^2}$. Evaluating all of the integrals expect the angular k integral:

$$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2}\frac{M^2}{(k_*+\mu)M + 2k_*\tilde{E}_m(k_*)}, \tag{21}$$

where k_* is the positive solution to $E_{\rm cm} - \tilde{E}_m(-k) - E_\pi(k) = 0$. We now see that the expression in the denominator is strictly non-zero for all k_*, so there are no divergences in the two particle phase space. Indeed, we needed an IR cutoff, and the parameter μ provides that cutoff.

The same is true for when $\mu \to -\mu$, but the IR cutoff argument is not quite so manifest. The denominator of the phase space integral is now

$$(k_*-\mu)M + 2k_*\breve{E}_m(k_*), \qquad \breve{E}_m(k) = \sqrt{(|\vec{p}\,|-\mu)^2 + m_e^2}. \tag{22}$$

If $E_{\rm cm} > \sqrt{m_e^2+\mu^2}$, then k_* is single-valued, and the denominator is strictly positive. There is a critical energy $E_{\rm critical}$ which is the smallest allowed value of $\breve{E}_m(-k)+E_\pi(k)$. At $E_{\rm critical}$, the phase space diverges linearly with k_*. For the case that $E_{\rm critical} < E_{\rm cm} < \sqrt{m_e^2+\mu^2}$, k_* is multi-valued but the denominator is strictly non-zero. (For multi-valued k_*, you can check that the phase space is the sum of the absolute values of the various integrands, so the fact that the denominator goes negative is not a problem.)

All we need to check is that for any realistic scattering, $E_{\rm cm} > E_{\rm critical}$. In the case of pseudo-Compton scattering with a incoming photon of energy ω, the center-of-mass energy is

$$E_{\rm cm} = \omega + \sqrt{(\omega\pm\mu)^2+m_e^2}. \tag{23}$$

For the plus sign case (corresponding to an incoming left-handed electron), $E_{\rm cm}$ is bounded below by $\sqrt{m_e^2+\mu^2}$, which is greater than $E_{\rm critical}$. For the minus sign case (corresponding to an incoming right-handed electron), $E_{\rm cm}$ is also bounded below by $\sqrt{m_e^2+\mu^2}$ if m_e is strictly greater than zero. And if m_e were exactly zero, then the axial $U(1)$ rephasing would allow us to remove the coupling between the electron and the π field. So in all cases, the phase space is finite (albeit large) for small $E_{\rm cm}$.

Now we can look at the cross-section for pseudo-Compton scattering to see whether the μ parameter makes it well behaved. Unfortunately, this is easier said than done. Because of the modification to the quadratic piece of the Lagrangian, the new electron propagator is

$$\left(\not k - m + \mu\gamma^0\gamma^5\right)^{-1} = \frac{\not k + m + \mu\gamma^0\gamma^5}{2}\left(\frac{1+\frac{\vec k\cdot\vec\gamma}{|\vec k|}\gamma^0\gamma^5}{\omega^2-(|\vec k|+\mu)^2-m^2} + \frac{1-\frac{\vec k\cdot\vec\gamma}{|\vec k|}\gamma^0\gamma^5}{\omega^2-(|\vec k|-\mu)^2-m^2}\right), \tag{24}$$

where $k=(\omega,\vec k)$. For the electron, there are two polarizations, but each has a different dispersion relation, so it is usually inconvenient to use the sum-over-polarization trick. To find the wave functions it is necessary to use the helicity basis to define spinors

$$\vec p\cdot\vec\sigma\,\xi_\pm(p) = \pm\xi_\pm(p), \tag{25}$$

where the normalization condition is $\xi^\dagger\xi = 1$. The wave functions and normalizations for the various helicity states are summarized in the following table.

	On-shell condition	Wave function	$u(k)\bar u(k)$ or $v(k)\bar v(k)$
e_L^-	$\omega^2 = (\lvert\vec k\rvert^2+\mu)^2+m^2$	$u_L(k) = \begin{pmatrix} +\sqrt{k\cdot\sigma+\mu}\;\xi_-(k) \\ +\sqrt{k\cdot\bar\sigma-\mu}\;\xi_-(k)\end{pmatrix}$	$\frac{\not k+m+\mu\gamma^0\gamma^5}{2}\left(1+\frac{\vec k\cdot\vec\gamma}{\lvert\vec k\rvert}\gamma^0\gamma^5\right)$
e_R^+	$\omega^2 = (\lvert\vec k\rvert^2-\mu)^2+m^2$	$v_R(k) = \begin{pmatrix} +\sqrt{k\cdot\sigma-\mu}\;\xi_-(k) \\ -\sqrt{k\cdot\bar\sigma+\mu}\;\xi_-(k)\end{pmatrix}$	$\frac{\not k-m-\mu\gamma^0\gamma^5}{2}\left(1+\frac{\vec k\cdot\vec\gamma}{\lvert\vec k\rvert}\gamma^0\gamma^5\right)$
e_R^-	$\omega^2 = (\lvert\vec k\rvert^2-\mu)^2+m^2$	$u_R(k) = \begin{pmatrix} +\sqrt{k\cdot\sigma+\mu}\;\xi_+(k) \\ +\sqrt{k\cdot\bar\sigma-\mu}\;\xi_+(k)\end{pmatrix}$	$\frac{\not k+m+\mu\gamma^0\gamma^5}{2}\left(1-\frac{\vec k\cdot\vec\gamma}{\lvert\vec k\rvert}\gamma^0\gamma^5\right)$
e_L^+	$\omega^2 = (\lvert\vec k\rvert^2+\mu)^2+m^2$	$v_L(k) = \begin{pmatrix} +\sqrt{k\cdot\sigma-\mu}\;\xi_+(k) \\ -\sqrt{k\cdot\bar\sigma+\mu}\;\xi_+(k)\end{pmatrix}$	$\frac{\not k-m-\mu\gamma^0\gamma^5}{2}\left(1-\frac{\vec k\cdot\vec\gamma}{\lvert\vec k\rvert}\gamma^0\gamma^5\right)$

(26)

In principle, the calculation is straight-forward and may be carried out at a later date. Our expectation is that the amplitude for pseudo-Compton scattering is well-behaved, and that the expression for the total cross-section is also well-behaved. (You might worry that we might have the same situation as in equation (16), where the divergence is worse than expected. Though we cannot rule out that possibility, it seems highly unlikely. Also, we may have made a calculational error in equation (16), such that the naïve expectation is actually correct.)

6 Prospects

We have seen in a very simple case that the couplings between the Standard Model and the π field of the form

$$\Delta\mathcal{L} = \frac{1}{F}\bar{\Psi}\gamma^5\vec{\gamma}\Psi \cdot \vec{\nabla}\pi + \mu\bar{\Psi}\gamma^0\gamma^5\Psi \tag{27}$$

are intimately linked. For consistency reasons, we cannot ignore the modification to the fermion dispersion relation when calculating scattering cross-sections involving π-ons. It would be amusing (though unlikely) to find the reverse case, where a modified fermion dispersion relation forced a couplings to some pseudo-scalar with a $\omega \sim k^2/M$ dispersion relation. In the theory of ghost condensation, the two effects are inseparable because of residual space diffeomorphisms, so there is a natural explanation for why they come paired.

There are other effects involving π-ons that have a similar pairing. It is possible (in fact, necessary because of graviton loops) to couple π to the electron momentum density, in which case you also have to change the speed of light (*i.e.* maximum attainable velocity) of the fermions. It would be interesting to see whether such couplings have a natural pairing in terms of insuring finite cross-sections. If nothing else, it is interesting to see the difficulties in doing calculations involving non-relativistic fields within a relativistic framework.

References

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[3] J. Thaler. *Spontaneous Time Diffeomorphism Breaking.* `http://www.jthaler.net/physics/notes/TimeDiffeoBreak.pdf`.