Classical and Quantum Mechanics on $SU(2)$

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1 Introduction

In this presentation, we will discuss the classical and quantum mechanics of the $SU(2)$ symmetry group. A group is any set of objects that has a defined multiplication and multiplicative inverse. The study of dynamics on groups is closely related to the study of group manifolds.

It turns out that $SU(2)$ has an interpretation as the motion of a top. Thus, a successful quantization of $SU(2)$ would imply a quantization of the top.

First, we will discuss what the $SU(2)$ symmetry group is. Then, we will discuss how to describe classical motion on a group manifold in general and for $SU(2)$ in particular. Finally, we will see how to quantize $SU(2)$ and arrive at a description of the quantum states of a top.

2 What is $SU(2)$?

$SU(2)$ is the group of *S*pecial *U*nitary 2x2 matrices. *Unitary* means that $U^\dagger U = 1$. By taking the determinant of both sides, we find that $|\det U| = 1$. *Special* imposes the additional condition that $\det U = 1$.

It is often convenient to express matrices in exponential form, $U = e^{iH}$. For U to be unitary, H must be Hermitian ($H^\dagger = H$), and for U to be special, H must be traceless. (These follows from the fact that $(e^{iH})^\dagger = e^{-iH^\dagger}$ and $\det e^A = e^{\mathrm{tr} A}$.) Thus, we say that $SU(2)$ is *generated* by the traceless Hermitian 2x2 matrices.

A convenient basis for the traceless Hermitian 2x2 matrices are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Actually, a bit more convenient are the matrices $t_i = \frac{\sigma_i}{2}$ because they satisfy the familiar commutation relation:

$$[t_a, t_b] = i\epsilon_{abc} t_c$$

We can write an arbitrary element of SU(2) as:

$$U = e^{i\varphi^1 \frac{\sigma_1}{2} + i\varphi^2 \frac{\sigma_2}{2} + i\varphi^3 \frac{\sigma_3}{2}}$$

or more compactly:

$$U = e^{i\varphi \hat{n}\cdot\frac{\vec{\sigma}}{2}}$$

$$\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3) \quad \varphi = |\vec{\varphi}| \quad \hat{n} = \frac{\vec{\varphi}}{\varphi} \quad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

By expanding the exponential and noting that $(\hat{n}\cdot\vec{\sigma})^2 = 1$, we also find that:

$$U = \cos\frac{\varphi}{2} + i\hat{n}\cdot\vec{\sigma}\sin\frac{\varphi}{2}$$

Though this parametrization of $SU(2)$ is helpful when we go to the quantization of the top, it doesn't really help us to see that $SU(2)$ has any real relation to the top. Let us look at a different but entirely equivalent parametrization:

$$U = e^{i\phi\frac{\sigma_3}{2}} e^{i\theta\frac{\sigma_2}{2}} e^{i\psi\frac{\sigma_3}{2}}$$

Here, ϕ, θ and ψ represent the Euler angles. (Note that the Pauli matrices do not commute so we cannot easily express this as a single exponential.)

3 Classical Dynamics on Group Manifolds

In classical mechanics, the key to determine the equations of motion is the Lagrangian. To understand motion on a group manifold, we need to find a formula for the Lagrangian given a time dependent group element.

Consider free particle motion in the plane. The Lagrangian is given by:

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2$$

We could also write this in terms of the column matrix:

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$L = \frac{1}{2}\dot{X}^T\dot{X}$$

For free motion on group manifolds, the Lagrangian is very similar:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{U}^\dagger\dot{U})$$

To see why the trace is necessary (and even intuitive), consider the expression:

$$\mathrm{Tr}(A^\dagger A) = \mathrm{Tr}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

Let's figure out the Lagrangian for $SU(2)$ in terms of Euler angles. If we explicitly multiply out the generic expression for U, take a time derivative, and evaluate L, we find:

$$L = \frac{1}{4}\dot{\phi}^2 + \frac{1}{4}\dot{\theta}^2 + \frac{1}{4}\dot{\psi}^2 + \frac{1}{2}\dot{\phi}\dot{\psi}\cos\theta$$

This is the Lagrangian for a top in Euler angles. The factor of one-half difference doesn't change the dynamics, but it makes things work out better when we go to quantization. We could have found the exact answer if we used the matrices $\frac{\sigma_i}{\sqrt{2}}$.

As an interesting side note, if we use the parametrization

$$(x^1, x^2, x^3) = \hat{n}\sin\frac{\varphi}{2}$$

$$x^4 = \cos\frac{\varphi}{2}$$

then the Lagrangian describes free particle motion on S_3, the surface of a four-dimensional sphere.

There are some interesting symmetries of the Lagrangian of $SU(2)$ (or indeed $SU(N)$ in general). Let V_L be a time-independent element of $SU(2)$ and consider the transformation $U(t) \to V_L U(t)$. This preserves the Lagrangian because:

$$L' = \frac{1}{2}\text{Tr}(\dot{U}^\dagger V_L^\dagger V_L \dot{U}) = \frac{1}{2}\text{Tr}(\dot{U}^\dagger \dot{U})$$

Similarly, for the transformation $U(t) \to U(t)V_R$,

$$L' = \frac{1}{2}\text{Tr}(V_R^\dagger \dot{U}^\dagger \dot{U} V_R) = \frac{1}{2}\text{Tr}(\dot{U}^\dagger \dot{U} V_R V_R^\dagger) = \frac{1}{2}\text{Tr}(\dot{U}^\dagger \dot{U})$$

(We have used the fact that $VV^\dagger = V^\dagger V = 1$ and $\text{Tr}(AB) = \text{Tr}(BA)$.)

There is a theorem due to Noether that every symmetry of the Lagrangian has an associated conserved quantity. We won't have to actually use Noether's Theorem, but we should expect to find two conserved quantities for motion on $SU(N)$.

It turns out that the conserved quantities show up in the equation of motion. Using the variational method for the action, consider an infinitesimal variation by right multiplication $U \to Ue^{ih} = U(1 + ih)$, where h is an infinitesimal Hermitian matrix.

$$\frac{d}{dt}U(1 + ih) = \dot{U} + i\dot{U}h + iU\dot{h}$$

$$\frac{d}{dt}(1 - ih)U^\dagger = \dot{U}^\dagger - ih\dot{U}^\dagger - i\dot{h}U^\dagger$$

$$L = \frac{1}{2}\text{Tr}(\dot{U}^\dagger\dot{U} + i\dot{U}^\dagger\dot{U}h + i\dot{U}^\dagger U\dot{h} - ih\dot{U}^\dagger\dot{U} - i\dot{h}U^\dagger\dot{U} + O(h^2)) = \frac{1}{2}\text{Tr}(\dot{U}^\dagger\dot{U}) - \text{Tr}(iU^\dagger\dot{U}\dot{h})$$

(We have used the fact that the trace is cyclic invariant and that $\dot{U}^\dagger U + U^\dagger\dot{U} = 0$, which follows from taking a time derivative of $U^\dagger U = 1$.)

Evaluating the action differential and using integration by parts:

$$\delta S = -\int \mathrm{Tr}(iU^\dagger \dot{U}\dot{h})dt = \int \mathrm{Tr}\left(\frac{d}{dt}(iU^\dagger \dot{U})h\right)dt$$

This yields the equation of motion:

$$\frac{d}{dt}(U^\dagger \dot{U}) = 0$$

So we have our first conserved quantity:

$$J = \frac{1}{i}U^\dagger \dot{U}$$

It is Hermitian ($J^\dagger = \frac{-1}{i}\dot{U}^\dagger U = \frac{1}{i}U^\dagger \dot{U}$) and traceless ($\mathrm{Tr}(\frac{1}{i}U^\dagger \dot{U}) = \mathrm{Tr}(\frac{1}{i}U^{-1}\dot{U}) = \mathrm{Tr}\left(\frac{d}{dt}\frac{1}{i}\ln U\right) = \mathrm{Tr}\left(\frac{d}{dt}H\right) = 0$). Similarly, if we consider an infinitesimal variation by left multiplication, we arrive at the other conserved quantity:

$$K = iU\dot{U}^\dagger$$

By various manipulations, it can be shown that the equations of motion defined by the conservation of J and K are in fact identical. Moreover, the Lagrangian can be rewritten as

$$L = \frac{1}{2}\mathrm{Tr}(J^2) = \frac{1}{2}\mathrm{Tr}(K^2)$$

which suggests that J and K are types of momenta. Because J and K are traceless and Hermitian, they can be expressed as a linear combination of our basis matrices:

$$J = J_1 t_1 + J_2 t_2 + J_3 t_3$$

$$K = K_1 t_1 + K_2 t_2 + K_3 t_3$$

J_i and K_i will become quantum mechanical operators when we quantize $SU(2)$.

4 Quantization of $SU(2)$

We will use the canonical quantization method to quantize $SU(2)$. Recall our Lagrangian in terms of Euler angles:

$$L = \frac{1}{4}\dot{\phi}^2 + \frac{1}{4}\dot{\theta}^2 + \frac{1}{4}\dot{\psi}^2 + \frac{1}{2}\dot{\phi}\dot{\psi}\cos\theta$$

To find the conjugate momenta, we use:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

This gives us:

$$\begin{aligned} p_\phi &= \tfrac{1}{2}(\dot{\phi} + \dot{\psi}\cos\theta) \\ p_\theta &= \tfrac{1}{2}\dot{\theta} \\ p_\psi &= \tfrac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta) \end{aligned}$$

There are non-trivial ordering issues that arise because these momenta do not commute. Additionally, there is a non-trivial integration measure for $SU(2)$ which affects the quantum mechanical momentum operators. If we make the naïve substitution $p_i = \frac{1}{i}\frac{\partial}{\partial q^i}$ and put all momentum operators on the far right, then we actually get the appropriate answers.

After a lot of algebra, we can write out the expressions for J_i and K_i in terms of Euler angles.

$$\begin{aligned}
J_1 &= \tfrac{1}{i}\left(\tfrac{\cos\psi}{\sin\theta}\tfrac{\partial}{\partial\phi} - \sin\psi\,\tfrac{\partial}{\partial\theta} - \tfrac{\cos\psi}{\tan\theta}\tfrac{\partial}{\partial\psi}\right) \\
J_2 &= \tfrac{1}{i}\left(\tfrac{\sin\psi}{\sin\theta}\tfrac{\partial}{\partial\phi} + \cos\psi\,\tfrac{\partial}{\partial\theta} - \tfrac{\sin\psi}{\tan\theta}\tfrac{\partial}{\partial\psi}\right) \\
J_3 &= \tfrac{1}{i}\tfrac{\partial}{\partial\psi}
\end{aligned}$$

$$\begin{aligned}
K_1 &= \tfrac{1}{i}\left(\tfrac{\cos\phi}{\sin\theta}\tfrac{\partial}{\partial\psi} - \sin\psi\,\tfrac{\partial}{\partial\theta} - \tfrac{\cos\phi}{\tan\theta}\tfrac{\partial}{\partial\phi}\right) \\
K_2 &= \tfrac{1}{i}\left(\tfrac{\sin\phi}{\sin\theta}\tfrac{\partial}{\partial\psi} + \cos\psi\,\tfrac{\partial}{\partial\theta} - \tfrac{\sin\phi}{\tan\theta}\tfrac{\partial}{\partial\phi}\right) \\
K_3 &= \tfrac{1}{i}\tfrac{\partial}{\partial\phi}
\end{aligned}$$

Working out the commutation relations, we find that:

$$\begin{aligned}
[J_a, J_b] &= i\epsilon_{abc}J_c \\
[K_a, K_b] &= i\epsilon_{abc}K_c \\
[J_a, K_b] &= 0
\end{aligned}$$

Thus, J and K act as nearly independent angular momenta. The reason why they aren't completely independent is that from the Lagrangian, we found that $\mathrm{Tr}(J^2) = \mathrm{Tr}(K^2)$, which implies that J^2 and K^2 share the same eigenvalues. From what we already know about angular momentum, this suggests that the eigenstates of the quantum top should be:

$$\vec{J}^2\,|j;m,k\rangle = \vec{K}^2\,|j;m,k\rangle = j(j+1)\,|j;m,k\rangle$$

$$J_3\,|j;m,k\rangle = m\,|j;m,k\rangle$$

$$K_3\,|j;m,k\rangle = k\,|j;m,k\rangle$$

Also, $J_1 + iJ_2$ and $K_1 + iK_2$ are the expected raising and lower operators.

Note that $\vec{J}^2$ here means $J_1^2 + J_2^2 + J_3^2$. To make a connection to $L = \frac{1}{2}\mathrm{Tr}(J^2)$, we can use the relation $\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$. It is easy to show that $L = \frac{1}{2}\mathrm{Tr}(J^2) = \frac{1}{4}\vec{J}^2$. Because this is free particle motion (and for other reasons, too) the Lagrangian equals the Hamiltonian, so the energy eigenstates are:

$$H\,|j;m,k\rangle = \frac{j(j+1)}{4}\,|j;m,k\rangle$$

Working out the complete eigenstates in Euler angles is not particularly fulfilling. (They are closely related to the spherical harmonics, as one might expect.) However, we can figure out the energy level degeneracy very easily because the energy levels only depend on j. There are $2j+1$ possible m values and $2j+1$ possible k values for each value of j, thus the total degeneracy is $(2j+1)^2$.

To find some simple "radial" eigenfunctions of the $SU(2)$ Hamiltonian, we need to find the appropriate expression for the Hamiltonian. The easiest form is to use the parametrization mentioned before:

$$U = e^{i\varphi\hat{n}\cdot\frac{\vec{\sigma}}{2}}$$

Evaluating the Lagrangian in this parametrization, we find:

$$L = \frac{1}{4}\dot{\varphi}^2 + \frac{1}{4}\dot{\hat{n}}^2 \sin^2 \frac{\varphi}{2}$$

By analogy to the case of a spherically symmetric potential we see that the Hamiltonian can be written as:

$$H = -\frac{1}{4}\frac{1}{\sin^2 \frac{\varphi}{2}}\frac{\partial}{\partial \varphi}\sin^2 \frac{\varphi}{2}\frac{\partial}{\partial \varphi} + \frac{1}{4}\frac{\hat{L}^2}{\sin^2 \frac{\varphi}{2}}$$

Considering eigenfunctions with no "angular" dependence, we find the solutions:

$$|j;0,0\rangle = \frac{\sin(j+\frac{1}{2})\varphi}{\sin \frac{\varphi}{2}}$$

which indeed have eigenvalues of $\frac{j(j+1)}{4}$.