# Two Lectures on $SU(N)$

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## 1 Introduction

In the last lecture, we learned about the dynamics of $SU(2)$. In these two lectures, we will extend the discussion to a more general group: $SU(N)$. In order to do so, we will need to learn more about the properties of groups, but the payoff will be insight into the relationship between fermions and bosons.

In the first lecture, we will start with a review of the last lecture. Then we will look at the theory of group representations and characters. In the second lecture, we will apply this theory to the study of quantum mechanics on $SU(N)$. We will see how this matrix model suggests a duality between fermions and bosons in quantum field theory.

***Lecture One***

## 2 A Look Back at $SU(2)$

The crucial realization about $SU(2)$ was that it had a large degree of symmetry. We found that the group Lagrangian

$$L = \frac{1}{2}\mathrm{Tr}(\dot{U}^\dagger \dot{U})$$

was invariant under left and right multiplication

$$U \to V_L U V_R$$

which led to two conserved quantities, $J$ and $K$, that acted like momenta. Furthermore, we found that we could express the Lagrangian in the form

$$L = \frac{1}{4}\vec{K}^2 = \frac{1}{4}\sum_\alpha K_\alpha^2$$

There was also another way to discuss the Lagrangian of $SU(2)$. If we introduced the parametrization

$$U = e^{i\varphi \hat{n}\cdot\frac{\vec{\sigma}}{2}}$$

then the Lagrangian took on the form

$$L = \frac{1}{4}\dot{\varphi}^2 + \frac{1}{4}\dot{\hat{n}}^2 \sin^2\frac{\varphi}{2}$$

We will generalize these two different but equivalent Lagrangians when we move to $SU(N)$.

At the end of the last presentation, I made a slight mistake. I introduced the function

$$\chi^{(j)}(\varphi) = \frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}}$$

and claimed this was an eigenstate of the Hamiltonian with no $J_3$ or $K_3$ component. Though $\chi$ is indeed an eigenstate of the Hamiltonian, $J_3|\chi\rangle \neq 0$ and $K_3|\chi\rangle \neq 0$. If we really wanted to express $\chi$ as an eigenstate and find its quantum numbers, we could introduce $\vec{L} = \vec{J} - \vec{K}$ and $\vec{M} = \vec{J} + \vec{K}$. But $\chi$ has another interpretation besides being an eigenstate of the Hamiltonian; $\chi$ is the *character* of the *representations* of $SU(2)$.

## 3 Representations of Groups

To talk about representations of groups, we should first understand the properties of groups. A group is collection of elements such that the following properties hold:

1. Identity: There is an element $e$ such that for every element $g$, $eg = ge = g$.
2. Inverses: For every element $g$ there is an element $g^{-1}$ such that $gg^{-1} = g^{-1}g = e$.
3. Closure: Given two elements $g_1$ and $g_2$, $g_1 g_2 = g_3$ is an element of the group.
4. Associativity: For any elements $g_1$, $g_2$ and $g_3$, $(g_1g_2)g_3 = g_1(g_2g_3)$.

The properties of any particular group are given by the multiplication rules. For example, if we have the collection of objects

$$g(\theta) = e^{i\theta}$$

the multiplication rule is

$$g(\theta_1)g(\theta_2) = g(\theta_1 + \theta_2).$$

From this multiplication rule, clearly $e = g(0)$ and $g(\theta)^{-1} = g(-\theta)$. By the way, this is the group $U(1)$.

The multiplication rule for $SU(2)$ isn't so simple to write down, but it still exists. A *representation* of a group is any collection of objects that preserves the group properties. In $SU(2)$ for example, we can associate to every 2x2 special unitary matrix $U_i$ a new object $\mathcal{U}_i$ such that:

$$\begin{aligned} U_e &\mapsto \mathcal{U}_e \\ (U_i)^{-1} &\mapsto (\mathcal{U}_i)^{-1} \\ U_1U_2 = U_3 &\mapsto \mathcal{U}_1\mathcal{U}_2 = \mathcal{U}_3 \end{aligned}$$

We often write $\mathcal{U} = \mathcal{U}(U)$ to emphasize the fact that $\mathcal{U}$ is a function of $U$. It can be shown that every representation of a group is equivalent to some higher dimensional unitary matrix representation, so we often say that we are looking for higher dimensional representations of $SU(2)$. Also, we want to look for *irreducible* representations, i.e. representations that cannot be written in matrix form as

$$\mathcal{U}^{reducible} = \begin{pmatrix} \mathcal{U}_1 & O \\ O & \mathcal{U}_2 \end{pmatrix}$$

There is an entirely equivalent way of talking about higher dimensional representations. We had the parametrization of $SU(2)$, $U(\varphi, \hat{n}) = e^{i\varphi\hat{n}\cdot\vec{t}}$ with the generators satisfying $[t_i, t_j] = i\epsilon_{ijk}t_k$. If we can find higher dimensional matrices such that

$$[T_i, T_j] = i\epsilon_{ijk}T_k$$

then our higher dimensional representation can be parametrized as

$$\mathcal{U}(\varphi, \hat{n}) = e^{i\varphi\hat{n}\cdot\vec{T}}$$

In other words, the $T_i$ matrices generate the new representation. It is a straightforward exercise to show that this parametrization preserves the group properties.

For $SU(2)$, the higher dimensional representations are in one-to-one correspondence with the quantum numbers $j$. We already knew this from the study of angular momentum and spin. The matrices $S_i$ are precisely $t_i$ (with a factor of $\hbar$). But the matrices $S_i$ have the same commutation relations as the more general matrices $J_i$. Therefore, if we associate $J_i$ (without the factor of $\hbar$) with $T_i$, we see right away that $T_i$ will be the generators for the higher dimensional representations of $SU(2)$.

## 4 Characters of Representations

One of the best ways to study the representations of group is by studying the characters of representations, because there is a one-to-one correspondence between the characters and the representations. The character is a function of the group elements, and the definition of the character is quite simple. For a given representation $R$, the character is:

$$\chi^{(R)}(U) = \mathrm{Tr}(\mathcal{U}(U))$$

Remember that $\mathcal{U}$ (the higher dimensional matrix) is a function of $U$ (the defining group element).

Let's look at three simple cases to get a feeling for the character. Take, for example, the trivial group representation where every element is associated with the number 1. (Yes, this does preserve group properties, though clearly it is not higher dimensional. The generators are $T_i = 0$, which satisfy the commutation relation.) The character is

$$\chi^{trivial} = \mathrm{Tr}(1) = 1$$

For the defining representation $\mathcal{U} = U$, the character is:

$$\chi^{defining}(U) = \mathrm{Tr}(U)$$

For any representation, we can evaluate the character at the identity element $U_e$.

$$\chi^{(R)}(U_e) = \text{Tr}(\mathcal{U}_e) = d_R$$

where $d_R$ is the dimension of the representation.

Let's find the characters of the representations of $SU(2)$. We can write any unitary matrix as

$$U = V^{-1}DV$$

where $V$ is a unitary matrix and $D$ is a diagonal matrix. The condition of special unitary implies that $D$ must take on the form

$$D = \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} = e^{i\varphi t_3}$$

(Expand the exponential and recall that $t_i^2 = \frac{1}{4}I$ to prove this last step.) Thus, a higher dimensional representation of U should look like:

$$\mathcal{U}(U) = \mathcal{U}(V^{-1}DV) = \mathcal{U}(V^{-1})\mathcal{U}(D)\mathcal{U}(V) = \mathcal{U}(V^{-1})\mathcal{U}(e^{i\varphi t_3})\mathcal{U}(V) = \mathcal{U}(V)^{-1}e^{i\varphi T_3}\mathcal{U}(V)$$

(Note that we have used the property that $\mathcal{U}$ maintains the group multiplication properties of $U$.) Evaluating the character:

$$\chi^{(R)}(U) = \text{Tr}(\mathcal{U}(U)) = \text{Tr}(\mathcal{U}(V)^{-1}e^{i\varphi T_3}\mathcal{U}(V)) = \text{Tr}(e^{i\varphi T_3})$$

We can express the trace in terms of the eigenvectors of $T_3$, $|jm\rangle$. (Recall that $T_3$ was virtually equivalent to the angular momenta $J_3$.)

$$\text{Tr}(e^{i\varphi T_3}) = \sum_{m=-j}^{j} \langle jm|e^{i\varphi T_3}|jm\rangle = \sum_{m=-j}^{j} \langle jm|jm\rangle e^{i\varphi m}$$

This is just a geometric series:

$$\sum_{m=-j}^{j} e^{i\varphi m} = \frac{e^{i(j+1)\varphi} - e^{i(-j)\varphi}}{e^{i\varphi} - 1} = \frac{e^{i(j+\frac{1}{2})\varphi} - e^{-i(j+\frac{1}{2})\varphi}}{e^{i\frac{\varphi}{2}} - e^{-i\frac{\varphi}{2}}} = \frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}}$$

Note that the character is only a function of $\varphi$ and *not* of $\hat{n}$. This turns out to be a very general properties of characters. To see this, we can write $U$ more explicitly as

$$U(\varphi, \hat{n}) = V(\hat{n})^{-1}D(\varphi)V(\hat{n})$$

Then evaluating the character:

$$\chi(U(\varphi, \hat{n})) = \chi(D(\varphi))$$

The matrix $D$ is just the eigenvalues of $U$, thus the character depends only on the eigenvalues of $U$ which in turn only depend of $\varphi$. This same argument can be used to show that the character is invariant under $U \to V^{-1}UV$.

Also note that the representations (given by the quantum number $j$) are in one-to-one correspondence with the characters, as expected. To find the dimension of the representation, we do a limiting procedure as $\varphi \to 0$ (this is the identity element):

$$d_j = \lim_{\varphi\to 0} \frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}} = \frac{j+\frac{1}{2}}{\frac{1}{2}} = 2j+1$$

This is precisely what we expect from our intuition about angular momenta.

## 5 Characters and Quantum Mechanics

How can we make a connection between characters as a property of a group and characters as eigenstates of the group Hamiltonian? One of the properties of eigenstates is that they are orthogonal and complete. Through a detailed study of group theory, one can prove that characters also satisfy orthogonality and completeness relations

$$\text{Orthogonality: } \int \chi^{R_I}(U)^\dagger \chi^{R_J}(U)\, dU = \delta_{IJ} \cdot (\text{a function of the group})$$

$$\text{Completeness: } \sum_I \chi^{R_I}(U_i)^\dagger \chi^{R_I}(U_j) = \delta_{ij} \cdot (\text{a function of the group})$$

The fact that the eigenstates of the Hamiltonian and the characters form the *same* orthogonal and complete set may seem a bit mysterious. It follows from the fact that the Hamiltonian is in fact a Casimir operator of the group. (A Casimir operator is an operator that commutes with all of the $K_\alpha$. It is similar to $\vec{J}^2$ in angular momenta.)

If this explanation isn't convincing, recall that the characters were invariant under $U \to V^{-1}UV$. When we were looking for the "radial" eigenfunctions of the Hamiltonian (i.e. functions of $\varphi$ and not $\hat{n}$) we were also looking for wavefunctions that were invariant under $U \to V^{-1}UV$. (This transformation basically represents a rotation of $\hat{n}$.)

In the next lecture, we will find the characters of $SU(N)$ in two different ways, one which follows from mainly from the group Lagrangian and one which follows more from group properties. The first relates $SU(N)$ to a fermion problem, whereas the second relates $SU(N)$ to a boson problem, suggesting a duality between fermions and bosons.

***Lecture Two***

## 6 A Note About Representations of $SU(N)$

When working with angular momentum (i.e. $SU(2)$), it was convenient to introduce raising and lowering operators $J_\pm = J_1 \pm J_2$. This allowed us to find all of the representations of $SU(2)$ by proving that any given representation had to have a highest weight (i.e. $m$ could only run from $-j$ to $j$.) In $SU(N)$ we can do the same thing, by introducing $N-1$ operators that are the analog of $J_3$, and $N^2 - N$ operators that are the analog of $J_\pm$. There will be a similar notion of highest weight in $SU(N)$ and the number of ways of lowering from the highest weight will give the dimension of the representation.

In the case of $SU(3)$, different representations correspond to different quark structures, and group theory is able to predict a lot about baryons, mesons, and resonances. For the quantum mechanics of $SU(N)$, though, finding the explicit form of the representations isn't all that important. In fact, much of the information about a representation is captured by the character of the representation, so we can bypass some of the group theory and go straight to the eigenstates of the group Hamiltonian.

# 7 Quantum Mechanics of $SU(N)$ (Method 1)

We start by finding the Lagrangian of $SU(N)$. Let $U = V^\dagger D V$ as before.

$$\dot{U} = \dot{V}^\dagger D V + V \dot{D} V + V^\dagger D \dot{V}$$

$$\dot{U}^\dagger = \dot{V}^\dagger D^\dagger V + V \dot{D}^\dagger V + V^\dagger D^\dagger \dot{V}$$

Using the fact that $D^\dagger D = 1$, $V^\dagger V = 1$ and the trace is cyclic invariant, we find that

$$L = \frac{1}{2}\mathrm{Tr}(\dot{U}^\dagger \dot{U}) = \frac{1}{2}\mathrm{Tr}(\dot{D}^\dagger \dot{D}) + \mathrm{Tr}(\dot{W} D^\dagger \dot{W} D) - \mathrm{Tr}(\dot{W}^2)$$

where $\dot{W} = V^\dagger \dot{V}$. (Note that $\dot{W}^\dagger = -\dot{W}$.) Because $U$ is special unitary, $D$ takes on the form

$$D = \begin{pmatrix} e^{i\varphi_1} & 0 & \cdots & 0 \\ 0 & e^{i\varphi_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\varphi_N} \end{pmatrix} = \mathrm{diag}[e^{i\varphi_i}]$$

$$\text{with } \sum_{i=1}^{N} \varphi_i = 0$$

The Lagrangian simplifies to:

$$L = \sum_i \frac{1}{2}\dot{\varphi}_i^2 + \sum_{ij} \dot{W}_{ij}(1 - e^{i\varphi_i} e^{-i\varphi_j}) = \sum_i \frac{1}{2}\dot{\varphi}_i^2 + \sum_{i>j} \dot{W}_{ij} \sin^2\left(\frac{\varphi_i - \varphi_j}{2}\right)$$

We will introduce the notation

$$\Delta = \sum_{i>j} \sin\left(\frac{\varphi_i - \varphi_j}{2}\right)$$

Note that $\Delta$ is anti-symmetric under an exchange of indices in $\varphi$. Because $\dot{W}_{ij}$ is a complex variable, the integration measure is $\Delta^2$. (If $\dot{W}_{ij}$ were real, the integration measure would be $\Delta$. Also $\Delta$ is real so $|\Delta|^2 = \Delta^2$.) We see that the Lagrangian separates into "radial" and "angular" variables just as in $SU(2)$. The "radial" Hamiltonian is thus:

$$H = -\sum_i \frac{1}{\Delta^2}\frac{\partial}{\partial \varphi_i}\Delta^2 \frac{\partial}{\partial \varphi_i} = -\sum_i \frac{1}{\Delta}\frac{\partial^2}{\partial \varphi_i^2}\Delta + \frac{1}{\Delta}\frac{\partial^2 \Delta}{\partial \varphi_i^2}$$

This last step follows by finding commutation relations between $\frac{1}{\Delta}$ and $\frac{\partial}{\partial \varphi_i}$. The second term turns out to just be a constant. (This follows from trigonometric identities involving cotangent.)

$$\sum_i \frac{1}{\Delta}\frac{\partial^2 \Delta}{\partial \varphi_i^2} = -\frac{N(N^2-1)}{12} \equiv -E_0$$

This may look like a very strange constant, but as we will see shortly it makes perfect sense.

When looking for "radial" eigenfunctions we are in effect looking for wavefunctions that are invariant under $U \to V^{-1}UV$. We already know that the trace is invariant under such a transformation. It turns out that we have a complete set of invariants of the form:

$$\rho_n = \mathrm{Tr}(U^n)$$

Because of completeness, the "radial" eigenfunctions will be functions of $\rho_n$. We will look more closely at $\rho_n$ when we turn to the second method. For the time being, all we need to know is that the trace is invariant under a permutation of $\varphi_i$, therefore the "radial" eigenfunctions will be invariant under a permutation of $\varphi_i$.

Because of the non-trivial integration measure, the inner product (for "radial" functions) is:

$$\langle \chi_1 | \chi_2 \rangle = \int \sum_i d\varphi_i \Delta^2 \, \chi_1(\varphi_i)^\dagger \chi_2(\varphi_i)$$

We can choose new wavefunctions that eliminate this integration measure:

$$\chi = \frac{1}{\Delta}\psi$$

The Hamiltonian now reads

$$\tilde{H} = -\sum_i \frac{\partial^2}{\partial \varphi_i^2} - E_0$$

Recall that $\varphi_i$ only runs from 0 to $2\pi$, so this is nothing but an $N$-body problem on a ring. Because $\chi$ was a symmetric wavefunction and $\Delta$ was anti-symmetric, this implies that $\psi$ must be an anti-symmetric wavefunction. In other words, $\psi$ represents $N$ *fermions* on a ring.

The wavefunction for a single fermion on a ring is

$$\psi_m(\varphi) = e^{im\varphi} \text{ with } E_m = m^2$$

Fermions cannot occupy the same energy level, so the lowest energy state would be (for $N$ odd):

$$E_0 = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} = \frac{N(N^2-1)}{12}$$

This is precisely the $E_0$ we found before! So the $-E_0$ in Hamiltonian renormalizes the energy such that the ground state of $SU(N)$ has energy 0.

The general eigenfunctions for $N$ fermions is given by the Slater determinant:

$$\psi(\varphi_i) = \begin{vmatrix} e^{im_1\varphi_1} & e^{im_2\varphi_1} & \cdots & e^{im_N\varphi_1} \\ e^{im_1\varphi_2} & e^{im_2\varphi_2} & \cdots & e^{im_N\varphi_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{im_1\varphi_N} & e^{im_2\varphi_N} & \cdots & e^{im_N\varphi_N} \end{vmatrix} \equiv [e^{im_1\varphi}, e^{im_2\varphi}, \cdots, e^{im_N\varphi}]$$

$$\text{with } m_1 > m_2 > \cdots > m_N$$

Clearly, this is antisymmetric under an exchange of indices because a determinant is antisymmetric under an exchange of rows. Because we are in $SU(N)$, $\varphi_1 + \varphi_2 + \cdots + \varphi_N = 0$, so we can factor out an $e^{im_N(\varphi_1+\varphi_2+\cdots+\varphi_N)}$ and rewrite $\psi$ as

$$\psi(\varphi) = [e^{i\ell_1\varphi}, e^{i\ell_2\varphi}, \cdots, e^{i\ell_n\varphi}]$$

$$\text{with } \ell_1 > \ell_2 > \cdots > \ell_N = 0$$

The character is thus:

$$\chi(\varphi) = \frac{1}{\Delta}[e^{i\ell_1\varphi}, e^{i\ell_2\varphi}, \cdots, e^{i\ell_n\varphi}]$$

We saw before that the character was in a one-to-one correspondence with the representations. Because the character depends on $\{\ell\}$, the representations should be defined by $\{\ell\}$. It turns out that it is slightly simpler to use new constants $\{\lambda\}$ to replace $\{\ell\}$.

$$\lambda_i = \ell_1 - N + i$$

Because $\ell_i$ was forced to be an integer, $\lambda_i$ satisfies

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N = 0$$

This enables us to draw a Young tableaux, which is a graphical way to describe the (irreducible) representations of $SU(N)$.

# 8 Quantum Mechanics of $SU(N)$ (Method 2)

There is an entirely equivalent way of find the characters, though at this point it may not be clear that they will match up to our results from Method 1. Remember that the group Lagrangian (and the Hamiltonian) could be written as:

$$L = \frac{1}{4}\vec{K}^2 = \frac{1}{4}\sum_\alpha K_\alpha^2$$

There is a clever way of writing $K_\alpha$ such that we don't need to introduce an arbitrary parametrization (such as the Euler angles we saw last time). Recall that

$$K = \frac{1}{i}U\dot{U}^\dagger$$

Making the canonical substitution $\dot{U}_{ij} \to \frac{1}{i}\frac{\partial}{\partial U_{ij}}$

$$K = \frac{1}{i}U_{ij}(\dot{U}^\dagger)_{jk} = \frac{1}{i}U_{ij}\overline{\dot{U}_{kj}} = \frac{1}{i}U_{ij}\overline{\frac{1}{i}\frac{\partial}{\partial U_{kj}}} = U_{ij}\frac{\partial}{\partial U_{kj}}$$

Define $\frac{\partial}{\partial U_{kj}} \equiv \left(\frac{\partial}{\partial U}\right)_{jk}$. Taking the $t^\alpha$ component of $K$, we find:

$$K^\alpha = \mathrm{Tr}\left(t^\alpha U \frac{\partial}{\partial U}\right)$$

This formula makes it clear that $K^\alpha$ has the following action on functions of $U$:

$$K_\alpha f(U) = f(t^\alpha U)$$

In particular:

$$K_\alpha(U_{uv}) = t^\alpha_{uw}U_{wv}$$

The $N^2-1$ generating matrices $t^\alpha$ satisfy an identity that will be useful.

$$\sum_\alpha t^\alpha_{uv} t^\alpha_{xy} = \frac{1}{2}\left(\delta_{uy}\delta_{vx} - \frac{1}{N}\delta_{uv}\delta_{xy}\right)$$

Recall that the character is invariant under a permutation of $\varphi_i$, and that $\rho_n = \mathrm{Tr}(U^n)$ form a complete set of invariants. We might expect that $\rho_n$ would be the characters themselves. Clearly $\rho_0$ (a constant) is an eigenstate of eigenvalue 0. Let's evaluate the Hamiltonian on $\rho_1$ using $K_\alpha$. (We are discarding a factor of 8 for simplicity.)

$$\begin{aligned} H(\rho_1) &= 2\sum_\alpha K_\alpha^2 \,\mathrm{Tr}(U) = 2\mathrm{Tr}(t^\alpha t^\alpha U) = 2 t^\alpha_{uv} t^\alpha_{vw} U_{wu} \\ &= \left(\delta_{uw}\delta_{vv} - \frac{1}{N}\delta_{uv}\delta_{vw}\right) U_{wu} = \left(N - \frac{1}{N}\right) U_{uu} = \left(N - \frac{1}{N}\right)\rho_1 \end{aligned}$$

So $\rho_1$ is indeed an eigenfunction of the Hamilton. (This is good, because we know that the character of the defining representation is $\mathrm{Tr}(U)$.)

For $\rho_2$, the story is a bit more complicated. For simplicity, we can actually use the (incorrect) identity $\sum_\alpha t^\alpha_{uv} t^\alpha_{xy} = \frac{1}{2}\delta_{uy}\delta_{vx}$, for though the eigenvalues will be wrong, the eigenfunctions will be correct.

$$\begin{aligned} H(\rho_2) &= 2\sum_\alpha K_\alpha^2 \, U_{uv}U_{vu} = 2\sum_\alpha K_\alpha (t^\alpha_{uw} U_{wv} U_{vu} + U_{uv} t^\alpha_{vw} U_{wu}) \\ &= 2\sum_\alpha t^\alpha_{uw} t^\alpha_{wx} U_{xv} U_{vu} + t^\alpha_{uw} U_{wv} t^\alpha_{vx} U_{xu} + U_{uv} t^\alpha_{vw} t^\alpha_{wx} U_{xu} + U_{uv} t^\alpha_{vw} t^\alpha_{wx} U_{xu} \\ &= \delta_{ux}\delta_{ww} U_{xv} U_{vu} + \delta_{ux}\delta_{wv} U_{wv} U_{xu} + \delta_{vx}\delta_{ww} U_{uv} U_{xu} + \delta_{vx}\delta_{ww} U_{uv} U_{xu} \\ &= N\rho_2 + (\rho_1)^2 + (\rho_1)^2 + N\rho_2 = 2(N\rho_2 + (\rho_1)^2) \end{aligned}$$

So $\rho_2$ isn't an eigenstate. Still, we can evaluate $H((\rho_1)^2) = 2(N(\rho_1)^2 + \rho_2)$. This suggests that linear combinations of $\rho_2$ and $(\rho_1)^2$ would be eigenstates. Indeed, we find the following eigenstates:

$$\frac{1}{2}\left(\rho_2 + (\rho_1)^2\right) \text{ and } \frac{1}{2}\left(\rho_2 - (\rho_1)^2\right)$$

We can continue on in this fashion to find all of the characters of $SU(N)$. And explicit formula involves Schur polynomials, and as expected, they depend on a series of constants $\{\lambda\}$, which shows the connection between Method 1, Method 2, and representation theory. In particular:

$$\begin{aligned} \chi_{\{1\}} &= \rho_1 \\ \chi_{\{2\}} &= \frac{1}{2}\left(\rho_2 + (\rho_1)^2\right) \\ \chi_{\{1,1\}} &= \frac{1}{2}\left(\rho_2 - (\rho_1)^2\right) \\ \chi_{\{3\}} &= \frac{1}{6}\left(2\rho_3 + 3\rho_2\rho_1 + (\rho_1)^3\right) \\ \chi_{\{2,1\}} &= \frac{1}{6}\left(\rho_3 - (\rho_1)^3\right) \\ \chi_{\{1,1,1\}} &= \frac{1}{6}\left(2\rho_3 - 3\rho_2\rho_1 + (\rho_1)^3\right) \\ &\vdots \end{aligned}$$

Given a particular value of $N$, it is simple to show the equivalence of $\chi$ from Methods 1 and 2.

## 9 From Matrix Models to Quantum Field Theory

To study the quantum mechanics of $SU(N)$ we didn't necessarily have to learn about group representations and characters. Method 1 followed quite simply from canonical quantum mechanics. Without group theory, though, we might never have had the insight to use the $K_\alpha$ form of the Hamiltonian to act on $\rho_n$.

In Method 1, we transformed the problem of Quantum Mechanics of $SU(N)$ into an $N$-body fermion problem. To study the $N$-body fermion problem further, we could introduce "second quantization," an unfortunate name for a derivation of quantum field theory.

But Method 2 offers us a second way to approach quantum field theory. Recall the definition of $\rho_n$:

$$\rho_n = \mathrm{Tr}(U^n) = \sum_i e^{in\varphi_i}$$

We can Fourier transform this to get:

$$\rho(\varphi) = \frac{1}{2\pi} \sum_n \rho_n e^{-in\varphi} = \sum_i \delta(\varphi - \varphi_i)$$

$\rho(\varphi)$ is a bosonic density field, and we can use $\rho(\varphi)$ to develop a quantum field theory of bosons. This is known as collective field theory.

The connection is clear. The extension of $SU(N)$ has an interpretation as both a quantum field theory of fermions and a quantum field theory of bosons. Therefore there must be a way within quantum field theory to "transform" fermions into bosons. This is known as *bosonization* and is one of the important results that comes out of quantum field theory.