Integrable Systems and Matrix Models

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1 Overview

When we looked at $SU(2)$ in June, we wanted to know whether a matrix model could have any connection to a real physical system. Indeed, we found recognizable properties of a top in the classical and quantum mechanics of $SU(2)$.

When we looked at $SU(N)$ in July, we wanted to know whether the group properties of matrix models could help us solve physical systems. Indeed, the idea of group representations and characters proved to be helpful in understanding the quantum mechanics of $SU(N)$.

Now in August, we want to know whether matrix models can help us to solve physical systems that would otherwise be difficult to solve in standard classical or quantum mechanics. In this presentation, we will discuss some general theory of integrable systems and show how matrices make it easier to solve a specific system: the Calogero model.

2 Integrable Systems

An integrable system is a system that has an integral of motion (i.e. conserved quantity) for each degree of freedom. The simplest example is a free particle in N dimensions.

$$L = \frac{1}{2}m\dot{\vec{x}}^2$$

$$\vec{p} = m\dot{\vec{x}} \text{ with } \dot{\vec{p}} = 0$$

The N dimensional vector $\vec{p}$ represents the N conserved quantities.

As one might guess, in order for a system to be integrable, the integrals of motion have to be independent, meaning that they cannot be functions of the other integrals of motion. Also, the integrals of motion have to be compatible, which means that their Poisson bracket (or commutator if we are in quantum mechanics) must be zero:

$$\{I_k, I_l\}_{PB} = 0$$

A general property of integrable systems is that they can be solved by *quadratures*, meaning a finite combination of algebraic operations, inverses of functions, and integrals of functions. In other words, an integrable system is a system for which an exact time evolution can be obtained analytically. In the case of the free particle above:

$$\vec{x}(t) = \frac{\vec{p}}{m}t + \vec{x}(0)$$

Let me emphasize at the outset that the integrability of a system does *not* depend on whether we have it in a matrix model form or not. That is, if a system is not integrable in the standard classical or quantum sense, then it still won't be integrable in a matrix model. But matrices do make it easier to see the integrals of motion for systems that are indeed integrable but whose integrals of motion are "hidden."

3 Lax Pairs

The key to finding integrals of motion for matrix models is the *Lax pair*. If the equations of motion of a system are equivalent to the matrix equation:

$$i\dot{A} = [B, A] \text{ with } B^\dagger = B$$

then Lax showed that the system is integrable. (The standard letters are L and M, but in this talk, L is the Lagrangian and M is the matrix, so A and B are less confusing.)

The proof of this is straightforward. The Lax equation looks a lot like the time evolution of an operator in the Heisenberg picture of quantum mechanics. So:

$$A(t) = e^{-iBt}A(0)\,e^{iBt}$$

(If you don't believe this equation, plug it back into the Lax equation to check.) Because B is Hermitian, e^{iBt} is unitary, so we have:

$$A(t) = U^\dagger A(0)\,U$$

Unitary transformations do not change the eigenvalues of an matrix, so the eigenvalues of $A(t)$ are the same as the eigenvalues of $A(0)$, thus the eigenvalues are conserved quantities. More conveniently, we have the integrals of motion:

$$I_k = \mathrm{Tr}(A^k)$$

For NxN matrices, it can be shown that I_k are independent up to $k = N$. To show that I_k are compatable is very difficult and the proof depends on the nature of the original problem, but it turns out to be true for the model we are studying. So as long as we accept independence and compatibility, then if we can reduce a matrix equation of motion to a Lax pair, then we have an integrable system.

4 The Calogero Problem

The Calogero Model is a integrable system that was solved in the early 1970s. The fact that a relatively simple classical system took so long to solve suggests that the problem of solving integrable systems is non-trivial.

The Calogero Lagrangian is:

$$L = \frac{1}{2}\sum_a \dot{\varphi}_a^2 - \sum_{a<b} \frac{g^2}{(\varphi_a - \varphi_b)^2}$$

This is a one-dimensional N-body system where the particles interact pairwise via a $1/r^2$ potential. The $1/r^2$ potential is a specific case of the more general Weierstrass potentials. The g value indicates the strength of the interaction.

To solve the Calogero Langrangian in either classical or quantum mechanics is very difficult. At first glance, there is no reason to expect that there would be integrals of motion at all. Our goal is to show that from the point of view of matrix models, the Calogero Problem is easily *shown to be* solvable even if it is not easily solvable.

5 The Matrix Model for the Calogero Problem

Consider the set of NxN Hermitian matrices. Because $M^\dagger = M$, the Lagrangian is:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{M}^2)$$

This Lagrangian is invariant under $M \mapsto U^\dagger M\, U$ where U is a unitary matrix. Unlike $SU(N)$, however, this Lagrangian is *not* invariant under left or right multiplication alone. The conserved quantity related to the single symmetry of the Lagrangian is:

$$J = i[M, \dot{M}]$$

(This is in some sense the sum of the J and K conserved quantities from $SU(N)$.)

We are going to see that this matrix model supplimented with the condition that J takes on a certain value will give the Calogero Lagrangian. You may wonder why we are allowed to set J to a certain value. We are free to consider the subset of the Hermitian matrices that satisfy the J constraint because it is a closed set (i.e. under the equations of motion a matrix with a certain J value will transform to another matrix with the same J value because J is constant in time).

The following bit of algebra will show the equivalence of the matrix Lagrangian and the Calogero Lagrangian. Let $M = V^\dagger D\, V$, where V is unitary and D is diagonal.

$$\dot{M} = \dot{V}^\dagger D\, V + V^\dagger \dot{D}\, V + V^\dagger D\, \dot{V} = V^\dagger(\dot{D} + i[W, D])\, V$$

where $W = i\dot{V}V^\dagger$. Note that $W^\dagger = W$ so W is Hermitian. (The above equation makes sense even if D is not diagonal. Remember this equation, because we will need it when finding the Lax pair.)

The Lagrangian is now:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{D}^2 + i\dot{D}[W,D] + i[W,D]\dot{D} - [W,D]^2)$$

The middle terms are traceless:

$$\begin{aligned}\mathrm{Tr}(\dot{D}[W,D] + [W,D]\dot{D}) &= \mathrm{Tr}(\dot{D}WD - \dot{D}DW + WD\dot{D} - DW\dot{D}) \\ &= \mathrm{Tr}(D\dot{D}W - D\dot{D}W + D\dot{D}W - D\dot{D}W) = 0\end{aligned}$$

(We have used the fact that diagonal matrices commute. Also note that the trace of a commutator is always zero because of the cyclic invariance of the trace.) If we let

$$D = \mathrm{diag}[\varphi_1, \varphi_2, \ldots, \varphi_N]$$

then

$$\mathrm{Tr}(\dot{D}^2) = \sum_a \dot{\varphi}_a^2$$

To deal with the last term, let's look at J more closely.

$$J = i[M, \dot{M}] = iV^\dagger[D, \dot{D} + i[W,D]\,]V = -V^\dagger[D,[W,D]\,]V$$

$$(-VJV^\dagger)_{ab} = ([D,[W,D]\,])_{ab} = (\varphi_a - \varphi_b)[W,D]_{ab}$$

Putting this all together:

$$\mathrm{Tr}([W,D]^2) = \sum_{a\,b}[W,D]_{ab}[W,D]_{ba} = \sum_{a\,b}\frac{(-VJV^\dagger)_{ab}(-VJV^\dagger)_{ba}}{(\varphi_a - \varphi_b)^2} = 2\sum_{a<b}\frac{(VJV^\dagger)_{ab}(VJV^\dagger)_{ba}}{(\varphi_a - \varphi_b)^2}$$

(The last step follows because the result is symmetric under an exchange of a and b.)

Now our Lagrangian is of the form:

$$L = \frac{1}{2}\sum_a \dot{\varphi}_a^2 - \sum_{a<b}\frac{(VJV^\dagger)_{ab}(VJV^\dagger)_{ba}}{(\varphi_a - \varphi_b)^2}$$

This is almost in the form we want, except we have a pole whenever $a = b$ and there is also some "angular" dependence. To get the Calogero model, we need:

$$(VJV^\dagger)_{ab}(VJV^\dagger)_{ba} = g^2(1 - \delta_{ab})$$

We will see that this is indeed the case for:

$$J_{ab} = g(1 - \delta_{ab})$$

(i.e. a matrix with zeros on the diagonal and g everywhere else).

To show this, we will need the property of unitary matrices that the rows and columns are orthonormal. That is:

$$\sum_k \sum_l V_{ak}V^\dagger_{lb} = \delta_{ab}$$

Let's do the algebra. (We are summing over k, l, m and n.)

$$\begin{aligned}
(VJV^\dagger)_{ab}(VJV^\dagger)_{ba} &= g^2(V_{ak}(1-\delta_{kl})V^\dagger_{lb})(V_{bm}(1-\delta_{mn})V^\dagger_{na}) \\
&= g^2(V_{ak}V^\dagger_{lb} - V_{ak}V^\dagger_{kb})(V_{bm}V^\dagger_{na} - V_{bm}V^\dagger_{ma}) \\
&= g^2(V_{ak}V^\dagger_{lb} - \delta_{ab})(V_{bm}V^\dagger_{na} - \delta_{ba}) \\
&= g^2(V_{ak}V^\dagger_{lb}V_{bm}V^\dagger_{na} - V_{ak}V^\dagger_{lb}\delta_{ba} - \delta_{ab}V_{bm}V^\dagger_{na} + \delta_{ab}\delta_{ba}) \\
&= g^2(V_{ak}V^\dagger_{na}V_{bm}V^\dagger_{lb} - \delta_{ab}\delta_{ba} - \delta_{ab}\delta_{ba} + \delta_{ab}) \\
&= g^2(\delta_{aa}\delta_{bb} - \delta_{ab}) \\
&= g^2(1-\delta_{ab})
\end{aligned}$$

So we have arrived at our final result, that the Calogero model is equivalent to a Hermitian matrix Lagrangian:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{M}^2)$$

subject to the constraint:

$$J = g(1-\delta_{ab})$$

6 The Lax Pair for the Calogero Matrix Model

All that remains is to show that the equation of motion for the Calogero Matrix Model reduces to a Lax pair. Using the Lagrange equations of motion:

$$\frac{\partial L}{\partial M} = \frac{d}{dt}\frac{\partial L}{\partial \dot{M}}$$

Clearly $\frac{\partial L}{\partial M} = 0$. To find $\frac{\partial L}{\partial \dot{M}}$ we have to be a bit more careful.

$$\frac{\partial}{\partial \dot{M}_{ij}}\left(\frac{1}{2}\mathrm{Tr}(\dot{M}^2)\right) = \frac{\partial}{\partial \dot{M}_{ij}}\left(\frac{1}{2}\dot{M}_{ab}\dot{M}_{ba}\right) = \frac{1}{2}\delta_{ai}\delta_{bj}\dot{M}_{ba} + \frac{1}{2}\dot{M}_{ab}\delta_{bi}\delta_{aj} = \frac{1}{2}\dot{M}_{ji} + \frac{1}{2}\dot{M}_{ji} = \dot{M}_{ji}$$

Thus:

$$0 = \frac{d}{dt}(\dot{M}) = \ddot{M}$$

(Note that there are no traces involved.) Let's turn this equation of motion into a Lax pair.

$$M = V^\dagger D\, V$$

$$\dot{M} = V^\dagger(\dot{D} + i[W,D])\, V$$

Let $A = \dot{D} + i[W,D]$.

$$\ddot{M} = V^\dagger(\dot{A} + i[W,A])\, V = 0$$

Our equation of motion is now:

$$V^\dagger(\dot{A} + i[W,A])\, V = 0$$

$$\dot{A} + i[W,A] = 0$$

$$i\dot{A} = [W,A]$$

We showed earlier that $W^\dagger = W$, thus A and W form the Lax pair we were looking for.