Investigations in Matrix Models

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1 Abstract

In Classical Mechanics we study the time evolution of vectors. In Quantum Mechanics we study the time evolution of wavefunctions. Similarly, in Matrix Models we study the time evolution of matrices.

Compared to wavefunctions, matrices are easy to work with because they have a finite number of degrees of freedom. Still, matrices are powerful objects because they satisfy the mathematical properties of a group. By combining insight from Group Theory with equations from Classical and Quantum Mechanics, we hope to create Matrix Models that mirror or approximate complicated physical systems.

Specific systems investigated so far include free particle motion on a hypersphere, corresponding to special unitary 2x2 matrices; fermion motion on a ring, corresponding to higher dimensional special unitary matrices; and interacting particles on a line, corresponding to Hermitian matrices with appropriate boundary conditions. More complicated Matrix Models should have relevance to String Theory.

2 An Overview of Matrix Models

2.1 The Lagrangian

Newton's Second Law ($F = ma$) can be reformulated in terms of the *Lagrangian*, the difference between the kinetic and potential energies:

$$L = KE - PE$$

The Lagrangian is a function of the coordinates, the velocities, and sometimes the time:

$$L = L(x, \dot{x}; t)$$

The Lagrange equation of motion is:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

This is entirely equivalent to Newton's Second Law but is much easier to manipulate.

For example, a particle in a potential well has the following kinetic and potential energies:

$$KE = \tfrac{1}{2}m\dot{x}^2 \quad PE = V(x)$$

The Lagrangian is thus:

$$L = \tfrac{1}{2}m\dot{x}^2 - V(x)$$

Using the Euler-Lagrange equations:

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \equiv F$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \equiv ma$$

So the equation of motion is the familiar Newton's Second Law:

$$F = ma$$

2.2 The Matrix Lagrangian

The most intuitive form for the Lagrangian of a free matrix (i.e. a matrix not in a potential well) is:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{M}^\dagger \dot{M})$$

To see this, look at the case of a 2x2 matrix:

$$\dot{M} = \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} \quad \dot{M}^\dagger = \begin{pmatrix} \dot{\bar{a}} & \dot{\bar{c}} \\ \dot{\bar{b}} & \dot{\bar{d}} \end{pmatrix}$$

$$L = \frac{1}{2}\mathrm{Tr}(\dot{M}^\dagger \dot{M}) = \frac{1}{2}|\dot{a}|^2 + \frac{1}{2}|\dot{b}|^2 + \frac{1}{2}|\dot{c}|^2 + \frac{1}{2}|\dot{d}|^2$$

This is the kinetic energy for four particles in the complex plane, so the matrix Lagrangian reduces to a problem that we have seen before.

So if the Matrix Lagrangian is nothing new, why study it at all? The answer lies in study of conserved quantities.

2.3 Lagrangian Invariance and Conserved Quantities

Conservation laws are some of the most powerful tools in physics. We are all familiar with concepts such as conservation of energy, conservation of electric charge, and conservation of momenta. These conservation laws are specific examples of a more general phenomena.

E. Noether's Theorem tells us that any coordinate transformation that leaves the Lagrangian invariant has a corresponding conserved quantity. In other words, if we change the coordinate x to x' but the Lagrangian L is the same as the Lagrangian L', then there should be a quantity that is constant in time.

The simplest example is conservation of linear momentum. Consider a transformation that shifts the coordinates to the right by a distance ℓ:

$$x' = x + \ell$$

If this is a free particle, then the the original Lagrangian would be:

$$L = \tfrac{1}{2}m\dot{x}^2$$

Our Lagrangian in the new coordinate is:

$$L' = \tfrac{1}{2}m\dot{x}'^2 = \tfrac{1}{2}m\left(\tfrac{d}{dt}(x+a)\right)^2 = \tfrac{1}{2}m\dot{x}^2$$

We see right away that $L' = L$, thus we should expect that there should be a conserved quantity. Indeed there is. Going to the Lagrange equation of motion:

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}(m\dot{x})$$

So the equation of motion is:

$$\frac{d}{dt}(m\dot{x}) = 0$$

which means that $m\dot{x}$ is the conserved quantity. Another way of writing $m\dot{x}$ is mv which should look familiar as the linear momentum.

2.4 Matrices and Lagrangian Invariance

When a system has a conserved quantity (i.e. the Lagrangian is invariant under some coordinate transformation), we say that the system has a *symmetry*. Many physical systems have symmetries, but often they are difficult to spot. The advantage of Matrix Models is that the symmetries are usually quite obvious.

For example, consider the time independent unitary matrix transformation:

$$M' = U^\dagger M U$$

where $UU^\dagger = 1$ and $\dot{U} = 0$. The free matrix Lagrangian is now:

$$L' = \frac{1}{2}\mathrm{Tr}(\dot{M'}^\dagger \dot{M'}) = \frac{1}{2}\mathrm{Tr}(U^\dagger \dot{M}^\dagger U U^\dagger \dot{M} U) = \frac{1}{2}\mathrm{Tr}(\dot{M}^\dagger \dot{M})$$

(Here we have used the cyclic invariance of the trace.) We see that under the unitary matrix transformation $L' = L$, so we should expect a conserved quantity. Indeed, there is:

$$J = i(M\dot{M} - \dot{M}M) \equiv i[M, \dot{M}] \text{ and } \dot{J} = 0$$

Thus, if we can find a physical system that is equivalent to a free matrix model, then we can use the conserved quantity J to help understand the system. Matrix Models with potentials may also have conserved quantities depending on the form of the potential.

2.5 Additional Advantages of Matrix Models

Non-singular matrices (i.e. matrices with non-zero determinant) form a *group*. A group is one of the simplest mathematical structures in that there is only one allowed operation (multiplication), but the properties of groups are quite important for physical systems. For example, when we move from Classical Mechanics to Quantum Mechanics, the wavefunctions of the Matrix Models correspond to the *characters* of the group.

Beyond just the group properties, most matrices can be rewritten in the following form:

$$M = e^{iH}$$

(Yes, the exponential of a matrix is well-defined, in case you were wondering.) These new matrices H are called the *generators* of the group and they form a *Lie Algebra*. Lie Algebras form the backbone of the operator formalism of Quantum Mechanics, so it is encouraging that Matrix Models contain such structures without any additional work.

3 Three Simple Matrix Models

This summer we have examined three matrix models in depth. Presentation notes from the first two models are available below. A presentation on the third model is forthcoming. Here is a summary of some findings.

3.1 Special Unitary 2x2 Matrices

Unitary means that $U^\dagger U = 1$. It is straightforward to show that such matrices satisfy $|\det U| = 1$. Special imposes the additional condition that $\det U = 1$. These matrices can be parametrized in the following form:

$$U(t) = e^{i\varphi(t)\hat{n}(t)\cdot\frac{\vec{\sigma}}{2}}$$

where φ is a "radial" distance, $\hat{n}$ is a unit vector, and $\vec{\sigma}$ are the Pauli matrices. With some algebra, it can be show that:

$$U(t) = \cos\tfrac{\varphi(t)}{2}\ I + \sin\tfrac{\varphi(t)}{2}\ \hat{n}(t)\cdot\vec{\sigma}$$

This form suggests introducing new variables:

$$\begin{aligned} x_1 &= \sin\tfrac{\varphi}{2}\hat{n}_1 \\ x_2 &= \sin\tfrac{\varphi}{2}\hat{n}_2 \\ x_3 &= \sin\tfrac{\varphi}{2}\hat{n}_3 \\ x_4 &= \cos\tfrac{\varphi}{2} \end{aligned}$$

These new variables now represent the motion of a free particle on a hypersphere:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

3.2 Special Unitary NxN Matrices

Another way to write a unitary matrix is as:

$$U = V^\dagger D V$$

where V is a new unitary matrix and D is a diagonal matrix:

$$D = \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_n})$$

where φ_i are real and $\sum_i \varphi_i = 0$.

It turns out that the φ_i variables represent the angular position of N non-interacting fermions on a ring. (To show this without some insight from group theory would be quite difficult.)

3.3 Hermitian NxN Matrices with Fixed "Angular Momentum"

Hermitian Matrices satisfy the property that $H^\dagger = H$. Thus, the Hermitian Lagrangian is a bit simpler then the general Matrix Lagrangian:

$$L = \frac{1}{2}\mathrm{Tr}(\dot{H}^2)$$

As we saw before, there was a conserved quantity in all simple free matrix models:

$$J = i[H, \dot{H}]$$

Because this is a conserved quantity, we can set it to whatever value we wish. It turns out that if we let

$$J_{ab} = k(1 - \delta_{ab})$$

(i.e. a matrix with zero along the diagonal and k everywhere else) then the Lagrangian takes on the form:

$$L = \sum_i \frac{1}{2}\dot{\varphi}_i^2 - \sum_{i<j} \frac{k^2}{(\varphi_i - \varphi_j)^2}$$

This is the Lagrangian for the Calogero system: N particles on a line interacting with a $1/r^2$ potential. The constant k indicates the strength of the interaction. (Unfortunately, the potentials for gravity and electrostatics is $1/r$. It can be shown that no simple matrix model can create a $1/r$ potential.)

4 Future Directions in Matrix Models

We plan to continue the study of Matrix Models by examining more complicated systems. One way to do so would be to study *Gauge symmetries* in Matrix Models. Roughly speaking, Gauge symmetries are time dependent symmetries as opposed to the time independent symmetries we have examined so far. Matrix Models plus Gauge symmetry should allow us to see a connection with Quantum Field Theory.

Another type of symmetry is *Supersymmetry*, which is a "hidden" symmetry in the sense that there is to date no experimental evidence for it. Matrix Models plus Supersymmetry should allow us to create Lagrangians that have some connection to String Theory.